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PROBLEMS

IN ILLUSTRATION OF THE PRINCIPLES

OF

PLANE COORDINATE GEOMETRY.

BY WILLIAM WALTON, M.A.,
TRINITY COLLEGE, CAMBRIDGE.

> χαλεπὸν μὴ παραδείγμασι χρώμενον ἱκανῶς ἐνδείκνυσθαί τι τῶν μειζόνων.
>
> Πλάτων.

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PREFACE.

WITHIN the last few years have appeared many valuable works on Coordinate Geometry, involving a variety of new and elegant methods of Algebraical investigation. Several of these treatises are enriched with numerous incidental problems, and to the solution of problems of a miscellaneous character some of them are principally devoted. The tendency of even detached and desultory problems is no doubt to sharpen the intellect of the learner, and to give him a true conception of the signification of corresponding and more abstract propositions: the classification however of problems by some law of affinity is, in my opinion, calculated not only to produce the same result in a still higher degree, but also to qualify him by an enlarged comparison for becoming skilful and judicious in effecting his own solutions. In this conviction has originated the composition of this work.

In preparing this collection of Problems for publication, I have been guided by two principles of arrangement, the one depending upon similarity of geometrical properties, the other upon analogy between methods of solution. In Algebraic Geometry the latter principle seems to be usually the more important, and, excepting where its application would have been disadvantageously at variance with the former, has been in this treatise adopted as the basis of classification.

The obligations under which I am placed to various authors, for so much of what is comprised in this volume, have been particularly acknowledged throughout its pages. The methods of solution, however, which I have given, are rarely even similar to those which are developed in the works where the geometrical properties were originally enunciated. The greater number or all of the solutions contained in many of these works are purely geometrical. The advantage, it is proper to add, which I have derived from the various Examination papers published in the University of Cambridge and its various Colleges, is very considerable.

Cambridge, 30*th January*, 1851.

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ERRATA.

Page	Line	Error.	Correction.
5,	5 from bottom,	$y = -x,\ y = x - 2a,$	$y = -2x,\ y = x - 3a.$
82,	7,	$4a,$	$2a.$
108,	7 from bottom,	$+ 6ay \cot\beta = 9a^2.$	$+ 2ay \tan\frac{\beta}{2} = a^2.$
109,	7,	+ or –.	– or +.
119,	5 from bottom,	$\frac{m}{k^2},$	$\frac{m}{k}.$

ADDENDUM.

Page 40. If $u = 0$, $v = 0$, be the equations to two straight lines, then, $x\cos\varepsilon + y\sin\varepsilon = \delta$ being the general type of the equation to a straight line, the two bisecting lines will be denoted by the equations $u \pm v = 0$. This useful proposition is due to Mr. Rutherford, *Annals of Philosophy* for May, 1843, p. 353.

It is intended to publish an additional table of Errata, should it be found desirable, in January 1852.

ALGEBRAIC GEOMETRY.

STRAIGHT LINE.

SECTION I.

Elementary Problems. Rectangular Axes.

1. THE distances of a certain point from the three lines

$$x\cos\alpha + y\sin\alpha = \delta,\ x\cos\alpha' + y\sin\alpha' = \delta',\ x\cos\alpha'' + y\sin\alpha'' = \delta'',$$

being respectively c, c', c'', to find the relation between the quantities α, α', α'', δ, δ', δ'', c, c', c''.

By the conditions of the problem, we have

$$\begin{aligned} x\cos\alpha + y\sin\alpha - \delta &= c, \\ x\cos\alpha' + y\sin\alpha' - \delta' &= c', \\ x\cos\alpha'' + y\sin\alpha'' - \delta'' &= c''; \end{aligned}$$

or, putting $\delta + c = \lambda$, $\delta' + c' = \lambda'$, $\delta'' + c'' = \lambda''$, and eliminating x and y by cross multiplication,

$$\lambda(\cos\alpha''\sin\alpha' - \cos\alpha'\sin\alpha'') + \lambda'(\cos\alpha\sin\alpha'' - \cos\alpha''\sin\alpha)$$
$$+ \lambda''(\cos\alpha'\sin\alpha - \cos\alpha\sin\alpha') = 0,$$

or

$$\frac{\sin(\alpha' - \alpha'')}{\lambda'\lambda''} + \frac{\sin(\alpha'' - \alpha)}{\lambda''\lambda} + \frac{\sin(\alpha - \alpha')}{\lambda\lambda'} = 0.$$

2. To find the coordinates of a point, the distances of which from two given straight lines are assigned.

Let X, Y, be the coordinates of the required point, and let the equations to the given straight lines be

$$\begin{aligned} lx + my &= \delta, \\ l'x + m'y &= \delta', \end{aligned}$$

(l, m) being the direction-cosines of the former, and (l', m') of the latter line.

Then, the algebraical symbols of the distances of the point (X, Y) from these lines being respectively λ, λ', we have

$$lX + mY - \delta = \lambda, \quad l'X + m'Y - \delta' = \lambda',$$

λ, λ', being positive or negative accordingly as the point (X, Y) is, relatively to the origin, on the more remote or on the nearer side of the corresponding lines.

From these two equations we get, for the coordinates of the required point,

$$X = \frac{(\delta + \lambda)\, m' - (\delta' + \lambda')\, m}{m'l - ml'},$$

$$Y = \frac{(\delta + \lambda)\, l' - (\delta' + \lambda')\, l}{l'm - lm'}.$$

3. To find the equation to a straight line such that, if perpendiculars be drawn to it from n proposed points, the sum of the perpendiculars on one side of it may be equal to the sum of the perpendiculars on the other.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3), ... (x_n, y_n), be the n points, and suppose the equation to the straight line to be

$$\lambda x + \mu y = \delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

λ, μ, being its direction-cosines.

Then the algebraical expressions for the distances of the n points from this line will be

$$\lambda x_1 + \mu y_1 - \delta, \ \lambda x_2 + \mu y_2 - \delta, \ \lambda x_3 + \mu y_3 - \delta, \ldots \lambda x_n + \mu y_n - \delta.$$

But, since the geometrical sums of the two opposite groups of perpendiculars are, by the condition of the problem, equal, it follows that the algebraical sum of all the perpendiculars must be zero. Hence

$$\lambda (x_1 + x_2 + x_3 + \ldots + x_n) + \mu (y_1 + y_2 + y_3 + \ldots + y_n) = n\delta \ldots (2).$$

Eliminating δ between (1) and (2), we get

$$\lambda (x_1 + x_2 + x_3 + \ldots + x_n - nx) + \mu (y_1 + y_2 + y_3 + \ldots + y_n - ny) = 0,$$

as the equation to a line satisfying the hypothesis.

Since the ratio of λ to μ is not determinate, it follows that there is an infinite number of such straight lines, all passing through a point of which the coordinates are

$$\frac{1}{n}(x_1 + x_2 + x_3 + \dots + x_n), \quad \frac{1}{n}(y_1 + y_2 + y_3 + \dots + y_n).$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 192, *troisième édition.*

4. To determine the position of a point in a given straight line, such that the difference between the squares of its distances from two given points may be equal to a given area.

Let the origin of coordinates coincide with one of the given points, and let the axis of x contain the other.

Let a be the abscissa of the latter point, and let the equation to the given line be

$$x \cos\alpha + y \sin\alpha = \delta \quad \dots\dots\dots\dots\dots\dots \quad (1).$$

The coordinates of the required point being (x, y), we have, by the condition of the problem, c^2 denoting the given area,

$$x^2 + y^2 = c^2 + (x - a)^2 + y^2,$$

or

$$x = \frac{a^2 + c^2}{2a} \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (2).$$

The ordinate y is known from (1) and (2), the intersection of the two lines, which they represent, being the required point.

Cor. Suppose the lines represented by (1) and (2) to be coincident: then

$$\sin\alpha = 0, \quad \frac{\cos\alpha}{\delta} = \frac{2a}{a^2 + c^2},$$

whence

$$\alpha = 0, \quad \delta = \frac{a^2 + c^2}{2a};$$

and the value of y cannot be ascertained.

If then we restrict α and δ respectively to the values 0, and $\frac{a^2 + c^2}{2a}$, the problem becomes indeterminate, every point in the given straight line satisfying the condition of the problem.

This example affords an instance of the degeneration of a determinate into an indeterminate problem, in consequence of our restricting the data by particular conditions, and belongs therefore, according to Playfair's interpretation of the ancient signification of the term, to a class of propositions called, by the old geometricians, *Porisms*. Playfair* says that "a *Porism* may be defined *a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions.*"

5. A straight line, inclined to the axis of x at an angle of 30°, cuts the positive axes of coordinates in A and B. To find the equation to a straight line bisecting AB and passing through the origin.

The required equation is

$$x = \sqrt{3} . y.$$

6. To find the equations to the four sides of a square, the coordinates of two of the angular points being (2, 3) and (3, 4).

The required equations are

$$x = 2, \quad y = 3, \quad x = 3, \quad y = 4.$$

7. To find the equation to a line which cuts the two lines

$$y = 2, \quad y = 3,$$

at distances 3, 4, respectively, from the axis of y.

The required equation is

$$x = y + 1.$$

8. To find the equations to the straight lines which pass through the intersection of the lines

$$y = 2x + 4, \quad y = 3x + 6,$$

and bisect the supplementary angles which they include.

The required equations are

$$y = (1 \pm \sqrt{2}) . (x + 2).$$

---

* *Transactions of the Royal Society of Edinburgh*, Vol. III. p. 170; 1794.

9. To find the distance of the point of intersection of the lines denoted by the equations

$$3x + 2y + 4 = 0, \quad 2x + 5y + 8 = 0,$$

from the line denoted by the equation

$$y = 5x + 6.$$

$$\text{Required distance} = \frac{62}{11\sqrt{(26)}}.$$

10. To find the tangent of the angle between the lines represented by the equations

$$2x + 3y + 4 = 0, \quad 3x + 4y + 5 = 0.$$

If ϕ denote the angle between them,

$$\tan\phi = \tfrac{1}{6}.$$

11. To find the distance of the origin of coordinates from the line, the equation of which is

$$\frac{x}{2} + \frac{y}{3} = 1.$$

$$\text{Required distance} = \frac{6}{\sqrt{(13)}}.$$

12. To find the equation to a line passing through the intersection of the lines

$$x = a, \quad x + y + a = 0,$$

and through the origin: also to the line passing through the point of intersection of the same lines and perpendicular to the latter of them.

The required equations are respectively

$$y = -x, \quad y = x - 2a.$$

13. To find the length of the perpendicular from the vertex upon the base of a triangle, the coordinates of the vertex being (3, 3), and those of the extremities of the base (0, 2), (2, 1).

$$\text{Required length} = \sqrt{5}.$$

14. To form the equation for determining the abscissa of a point, in the straight line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the distance of which from a given point (α, β) shall be δ, and thence to shew that there are in general two such points and that, in the particular case in which those two points coincide,

$$\delta^2 . (a^2 + b^2) = (a\beta + b\alpha - ab)^2.$$

15. To find the abscissa of a point in the axis of x, such that its distance from a point $(0, k')$ may exceed its distance from a point (h, k), by a given length c.

If x represent the required abscissa,

$$(c^2 - h^2)\, x = e^2 h + c\,\{e^4 - (c^2 - h^2)\, k'^2\}^{\frac{1}{2}},$$

where
$$e^2 = \tfrac{1}{2}(c^2 + k'^2 - h^2 - k^2).$$

Newton: *Arithmetica Universalis*, Prob. XI.

16. If $ABCDE$ is a regular pentagon; to find the equations to its several sides, AB being the axis of x, and a line through A, at right angles to AB, the axis of y. To find also the equation to the line joining E, B, and to the line joining A, C.

The equations to the lines AB, BC, CD, DE, EA, EB, AC, are, respectively,

$$y = 0, \quad y = \tan\frac{2\pi}{5}(x - a),$$

$$y = -\tan\frac{\pi}{5}(x - a) + 2a\sin\frac{\pi}{5}, \quad y = \tan\frac{\pi}{5} . x + 2a\sin\frac{\pi}{5},$$

$$y = -\tan\frac{2\pi}{5} . x, \quad y = -\tan\frac{\pi}{5}(x - a), \quad y = \tan\frac{\pi}{5} . x.$$

17. To prove that, if the distances of a point from two points (a, b), (a', b'), are c, c', respectively, the point will lie in a straight line, the intercepts of which on the coordinate axes are

$$\tfrac{1}{2}(a' + a) + \frac{(b'^2 - b^2) - (c'^2 - c^2)}{2\,(a' - a)},$$

and
$$\tfrac{1}{2}(b' + b) + \frac{(a'^2 - a^2) - (c'^2 - c^2)}{2\,(b' - b)}.$$

18. If the ordinates of a straight line, drawn perpendicularly to the axis of x, be all turned about their intersections with this axis, so as to make an angle θ with it; to shew that, the equation to the original line being

$$y = x \tan \alpha,$$

the angle ϕ between this line and the rectilinear locus of the extremities of the inclined ordinates, may be found from the equation

$$\tan \phi = \frac{\sin \alpha + \tan \alpha \, . \sin (\alpha - \theta)}{\cos \alpha + \tan \alpha \, . \cos (\alpha - \theta)};$$

and that the new line will coincide with the original one, provided that

$$\theta = \tfrac{1}{2}\pi + 2\alpha.$$

19. To prove that the equation

$$2y^2 - 3xy - 2x^2 - y + 2x = 0$$

represents two straight lines perpendicular to each other.

20. To shew that the equation

$$y^2 - xy - 2x^2 + 5x - y - 2 = 0$$

represents two straight lines inclined to each other at an angle $\tan^{-1} 3$.

21. To prove that the equation

$$y^2 - 2xy \, . \sec \alpha + x^2 = 0$$

represents two straight lines, including an angle α.

SECTION II.

Elementary Problems. Oblique Axes.

1. To determine the geometrical signification of the equation

$$4y^2 - 8xy + x^2 + 4y + 2x - 2 = 0.$$

Completing the square with regard to y, we have

$$(2y + 1 - 2x)^2 = 3 (1 - x)^2,$$

and therefore $2y + 1 - 2x = \pm\sqrt{3}\,(1 - x)$,

or $2y + (\pm\sqrt{3} - 2)\,x = \pm\sqrt{3} - 1$,

the equations to two straight lines.

2. To determine the geometrical signification of the equation

$$x^2 + y^2 + xy = 0.$$

Multiplying the equation by 4, we have

$$(x + 2y)^2 + 3x^2 = 0.$$

This equation cannot be true, unless each of the squares be separately equal to zero. Hence

$$x = 0,$$

and $$x + 2y = 0,$$

and therefore x and y must be simultaneously equal to zero. The proposed equation therefore represents the origin of coordinates.

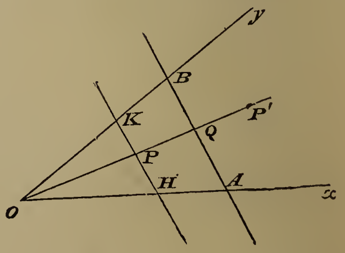
3. The axes of coordinates being inclined to each other at an angle of 60°, to determine the equation to a line parallel to a line, the equation of which is

$$x + y = 3a,$$

and at a distance from it equal to $\frac{1}{2}a\sqrt{3}$.

Let $OA = 3a = OB$; then AB is the line represented by the equation

$$x + y = 3a.$$



Draw OPQ at right angles to AB, and take $QP = \frac{1}{2}a\sqrt{3}$. Draw HPK parallel to AQB.

Then $PQ = AH \cos 30°$,

$$\tfrac{1}{2}a\sqrt{3} = \tfrac{1}{2}\sqrt{3}\,.\,HA,$$

$$HA = a = KB.$$

Hence $OH = 2a = OK$, and the equation to HK is therefore

$$x + y = 2a.$$

If we had taken a point P' on the other side of Q, so that $P'Q = \frac{1}{2}a\sqrt{3}$, we should evidently have got as the equation to the required line

$$x + y = 4a.$$

4. To determine the conditions that the equation

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0$$

may represent two parallel straight lines.

If the equation represent two parallel straight lines, its left-hand member must be resolvable into two factors of the forms

$$\lambda x + \mu y + \nu, \quad \lambda x + \mu y + \nu'.$$

Hence, identically,

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = (\lambda x + \mu y + \nu)(\lambda x + \mu y + \nu').$$

Equating the coefficients of x^2, y^2, $2xy$, x, y, on both sides, we have

$$\lambda^2 = a \text{.........}(1), \quad \mu^2 = b \text{.........}(2), \quad \lambda\mu = c \text{.........}(3),$$

$$\lambda(\nu + \nu') = 2a' \text{......}(4), \quad \mu(\nu + \nu') = 2b' \text{......}(5).$$

From (1) and (4), we have

$$\frac{a}{2a'} = \frac{\lambda}{\nu + \nu'},$$

and, from (3) and (5),

$$\frac{c}{2b'} = \frac{\lambda}{\nu + \nu'},$$

whence

$$\frac{a}{a'} = \frac{c}{b'} \text{..............................}(6).$$

Similarly, from (2) and (5), combined with (3) and (4), there is

$$\frac{b}{b'} = \frac{c}{a'} \text{..............................}(7).$$

The equations (6) and (7) constitute the required conditions.

5. To prove that the lines of which the equations are

$$y = 2x + 3, \quad y = 3x + 4, \quad y = 4x + 5,$$

all pass through one point.

6. To find the value of α that the line

$$y = \alpha x + 3$$

may pass through the intersection of the two lines

$$y = x + 1, \quad y = 2x + 2.$$

The required value of α is 3.

7. To find the equations to the two straight lines which, passing through the origin of coordinates, divide into three equal parts the distance between the points, in which the axes of coordinates are intersected by the line

$$x + y = 1.$$

The required equations are

$$2x = y \text{ and } 2y = x.$$

8. To find the equations to the diagonals of the parallelogram formed by the four lines

$$x = a, \quad x = a', \quad y = b, \quad y = b'.$$

The required equations are

$$(a' - a)\, y - (b' - b)\, x = a'b - ab',$$

and $$(a - a')\, y - (b' - b)\, x = ab - a'b'.$$

9. The equations to two straight lines referred to oblique axes, inclined to each other at an angle ω, being

$$y - mx = 0, \text{ and } my + x = 0,$$

to find the angle between them.

The angle between them is equal to

$$\tan^{-1}\left(\frac{m^2 + 1}{m^2 - 1} \tan \omega\right).$$

10. If the straight lines denoted by the equations

$$lx + my = \delta, \quad l'x + m'y = \delta',$$

be perpendicular to each other, to prove that, ω being the inclination of the axes,

$$ll' + mm' = (lm' + l'm) \cos \omega.$$

11. To prove that the group of straight lines defined by the equation

$$y - \frac{a'b - ab'}{a' - a} = \lambda \left(x - \frac{b - b'}{a - a'}\right),$$

where λ is arbitrary, are identical with the group of straight lines defined by the equation

$$y - ax - b = \mu (y - a'x - b'),$$

where μ is arbitrary.

12. What are the geometrical significations of the equations

$$x^2 + y^2 = 0, \quad xy = 0?$$

The former represents the origin, and the latter the axes of coordinates.

13. A straight line passes through a given point; to determine the magnitudes of its intercepts on the axes of coordinates, the product of the intercepts being given.

If a, b, be the coordinates of the given point, and c^2 the given product; then, m, n, representing the required intercepts,

$$m = \frac{c}{2b} \{c \pm (c^2 - 4ab)^{\frac{1}{2}}\},$$

$$n = \frac{c}{2a} \{c \mp (c^2 - 4ab)^{\frac{1}{2}}\}.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 141, *troisième édition.*

14. To determine the position of a straight line, which passes through a given point, and cuts two given straight lines so that its intercepted portion is of a given length.

If the two given lines be taken as axes of coordinates; then, ω denoting their included angle; h, k, the coordinates of the given point, and c the given length; the intercepts a, b, of the straight line on the axes of coordinates will be determined by the two equations

$$a^4 - 2(h + k\cos\omega)\,a^3 + (h^2 + k^2 - c^2 + 2hk\cos\omega)\,a^2 + 2c^2ha - c^2h^2 = 0,$$

$$b^4 - 2(k + h\cos\omega)\,b^3 + (k^2 + h^2 - c^2 + 2kh\cos\omega)\,b^2 + 2c^2kb - c^2k^2 = 0.$$

Newton: *Universal Arithmetic*, prob. 24.
Gergonne: *Annales de Mathématiques*, tom. 10, p. 205.

15. Through a given point in a given angle, equidistant from its two sides, to draw a straight line terminated at the two sides of the angle, so that this point may divide the straight line into two parts the sum of the squares of which shall be of a given value.

Let a, a, be the coordinates of the given point referred to the sides of the angle as axes; let ω denote the given angle. Then the equation to the required line will be

$$y - a = m(x - a),$$

m being determined by the equation

$$\left(m + \frac{1}{m} + \cos\omega\right)^2 = \frac{c^2}{a^2} + \cos^2\omega;$$

the four values of m corresponding to four different lines which satisfy the conditions of the problem.

"Ce problème est un de ceux qui furent proposés en 1819 au concours général des colléges royaux de Paris."

Gergonne: *Annales de Mathématiques*, tom. X. p. 73.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 142, *troisième édition.*

16. To ascertain the geometrical signification of the equation

$$y^2 - xy + \tfrac{5}{4}x^2 - 2x + 1 = 0.$$

It represents the point of intersection of two straight lines

$$x = 1, \quad y = \tfrac{1}{2}.$$

17. To determine the geometrical meaning of the equation

$$x^3 - 4a^2x = 0.$$

It represents three parallel straight lines, viz. the axis of y, and two equidistant lines on each side of it.

18. To interpret the equation

$$xy - 2ay + 3ax - 6a^2 = 0.$$

It represents two straight lines, one parallel to the axis of y, cutting the axis of x at a distance $+ 2a$ from the origin, and the other parallel to the axis of x, cutting the axis of y at a distance $- 3a$ from the origin.

19. To assign the geometrical signification of the equations

$$(x + y + c).(x - y + c) = 0, \quad (x + y + c)^2 + (x - y + c)^2 = 0.$$

The former equation denotes two straight lines, the intercepts of which on the axes of coordinates are respectively $(- c, - c)$ and $(- c, c)$. The latter equation denotes a point, viz. the point of intersection of the two lines.

20. To ascertain the geometrical signification of the equation

$$\left(\frac{x}{a} + 1\right)^2 + \frac{3}{2}\left(\frac{x}{a} + 1\right)\frac{y}{b} = \frac{y^2}{b^2}.$$

The equation represents two straight lines, the equations of which are

$$\frac{x}{a} + \frac{2y}{b} + 1 = 0, \quad \frac{x}{a} - \frac{y}{2b} + 1 = 0.$$

21. If the equation

$$ax^2 + by^2 + 2cxy = c'$$

represent two different straight lines; to prove that, either c' is zero and $c^2 > ab$, or c' is finite, and $c^2 = ab$.

22. To prove that the equation

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \cdot \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) = \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right)^2$$

represents two straight lines, and to find their equations.

The equations to the two lines are included in the single equation

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m},$$

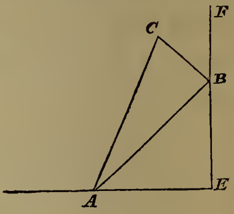
the two values of the ratio between l and m being determinable from the quadratic

$$l^2(b^2-\beta^2)+2lm\alpha\beta+m^2(a^2-\alpha^2)=0.$$

Section III.

Polar Equation.

1. AB and BC are two right lines perpendicular to each other; A is a fixed point, B moves along a given right line EF, and AB is to BC in a given ratio; to determine the locus of C.



From A draw AE at right angles to EF.

Let $\angle CAE=\theta$, $\angle CAB=\alpha$;

α being a constant quantity, because the ratio of AB to BC is constant.

Let $$AE=a,\quad AC=r.$$

Then $$r\cos\alpha=AB=a\sec(\theta-\alpha),$$

$$r=a\sec\alpha\sec(\theta-\alpha),$$

the equation to the locus of C, which is therefore a straight line.

2. To find the polar coordinates of the point of intersection of the straight lines represented by the equations

$$r=2a\sec(\theta-\tfrac{1}{2}\pi),\quad r=a\sec(\theta-\tfrac{1}{6}\pi),$$

and the angle between them.

The polar coordinates required are $(\frac{1}{2}\pi, 2a)$, and $\frac{1}{3}\pi$ is the angle between the two lines.

3. To find the polar equation to a straight line which passes through the two points (r_1, α_1), (r_2, α_2).

The required equation is

$$\frac{\sin(\alpha_2 - \alpha_1)}{r} + \frac{\sin(\theta - \alpha_2)}{r_1} + \frac{\sin(\alpha_1 - \theta)}{r_2} = 0.$$

4. If, from a point O taken arbitrarily in the plane of a triangle, straight lines be drawn to its summits A, B, C, and then, through the same point O, perpendiculars be drawn to the three lines OA, OB, OC; to prove that the intersection of these perpendiculars with the sides BC, CA, AB, respectively, will lie all three in one straight line.

If O being taken as pole, and the coordinates of A, B, C, be respectively (α_1, r_1), (α_2, r_2), (α_3, r_3); then, putting for brevity $\alpha_3 - \alpha_2 = \varepsilon_1$, $\alpha_1 - \alpha_3 = \varepsilon_2$, $\alpha_2 - \alpha_1 = \varepsilon_3$, we shall obtain for the equation to the line containing the three intersections,

$$\frac{2}{r}.\sin\varepsilon_1.\sin\varepsilon_2.\sin\varepsilon_3 + \frac{\sin 2\varepsilon_1}{r_1}.\cos(\alpha_1 - \theta) + \frac{\sin 2\varepsilon_2}{r_2}.\cos(\alpha_2 - \theta)$$
$$+ \frac{\sin 2\varepsilon_3}{r_3}.\cos(\alpha_3 - \theta) = 0.$$

Bobillier: *Gergonne*, *Annales de Mathématiques*, tom. XVIII. p. 185.

SECTION IV.

Rectilinear Loci.

1. To find the locus of a point equidistant from two given straight lines.

Let the equations to the given lines be

$$x\cos\alpha + y\sin\alpha = \delta,$$

and

$$x\cos\alpha' + y\sin\alpha' = \delta'.$$

Then x_1, y_1, being the coordinates of the point, its distances from the two lines will be

$$\pm(x_1\cos\alpha + y_1\sin\alpha - \delta),$$

and

$$\pm(x_1\cos\alpha' + y_1\sin\alpha' - \delta').$$

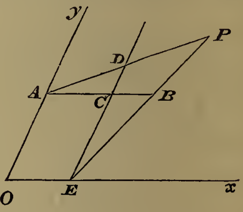
Since these distances are equal, we must have

$$x_1 \cos\alpha + y_1 \sin\alpha - \delta = \pm (x_1 \cos\alpha' + y_1 \sin\alpha' - \delta');$$

this result shews that the required locus is two straight lines at right angles to each other and passing through the point of intersection of the two given lines.

2. *ACB*, *DCE*, are two right lines of given lengths, *C* is the middle point of *AB*, and the line through *E* and *B* meets that through *A* and *D* in a point *P*; supposing *AB* to move parallel to itself, while *DE* remains fixed, to find the locus of *P*.

Let xEO, parallel to BA, be taken as the axis of x, and yAO, parallel to DE, as that of y.



Let

$$AC = a = BC, \quad CE = b, \quad DE = c.$$

Then the equation to AP, which passes through the points (O, b) and (a, c), is

$$ay - cx = b(a - x) \text{..................} (1).$$

The equation to EP, which passes through the points (a, O) and $(2a, b)$, is

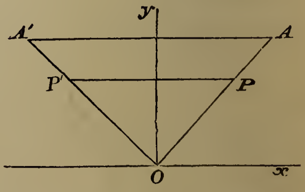
$$ay = b(x - a) \text{.....................} (2).$$

At the intersection P, of (1) and (2), we have, adding together the two equations,

$$2ay = cx,$$

the equation to the locus of P, which is therefore a straight line passing through O.

3. A straight line PP' is drawn parallel to the base AA' of a triangle AOA'. From P, P', are erected perpendiculars upon OA, OA', respectively; to find the locus of the intersection of these perpendiculars.



Take Ox, parallel to $A'A$, as the axis of x, and Oy, at right angles to $A'A$, as that of y. Let

$$\angle AOy = \varepsilon, \quad \angle A'Oy = \varepsilon';$$

then the equation to OA will be

$$x = y \tan \varepsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation to PP' will be of the form

$$c = y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The coordinates of P will therefore be

$$c \tan \varepsilon, \quad c,$$

and the equation to the perpendicular through P will be

$$y - c = -\tan \varepsilon . (x - c \tan \varepsilon) \ldots\ldots\ldots\ldots(3).$$

Similarly, putting $-\varepsilon'$ in place of ε, we shall have for the equation to the perpendicular through P',

$$y - c = \tan \varepsilon' (x + c \tan \varepsilon') \ldots\ldots\ldots\ldots(4).$$

At the intersection of (3) and (4),

$$x (\tan \varepsilon + \tan \varepsilon') = c (\tan^2 \varepsilon - \tan^2 \varepsilon'),$$

$$x = c (\tan \varepsilon - \tan \varepsilon'),$$

and therefore, from (3) or (4), eliminating c, we shall readily get, for the equation to the locus of the intersection of the two perpendiculars

$$y = \frac{x}{\tan (\varepsilon - \varepsilon')};$$

the required locus is therefore a straight line through O.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 124, *troisième édition*.

Lardner: *Algebraic Geometry*, vol. I. p. 29.

4. AB, CD, are two given parallel straight lines; through C and D are drawn two straight lines PCF, PDG, cutting AB in F, G, so that the ratio of AF to BG is invariable. To find the locus of the point P.

Let AB, produced indefinitely, be taken as the axis of x, and a line through A, at right angles to AB, as that of y.

Let (h, k), (h', k), be the coordinates of C, D, respectively; (x, y) those of P. Let $AB = c$. Also let

$$n . AF = n' . BG \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

n, n', being constant quantities.

The equation to *PF*, which passes through the two points *C*, *P*, will be

$$(k-y)\,x' - (h-x)\,y' = kx - hy \ldots\ldots\ldots\ldots(2).$$

Similarly, the equation to *PG*, which passes through the two points *D*, *P*, will be

$$(k-y)\,x' - (h'-x)\,y' = kx - h'y \ldots\ldots\ldots\ldots(3).$$

Putting $y'=0$ in (2) and in (3), we get, from (2),

$$AF = x' = \frac{kx-hy}{k-y},$$

and, from (3), $$AG = x' = \frac{kx-h'y}{k-y},$$

whence $$BG = c - \frac{kx-h'y}{k-y}.$$

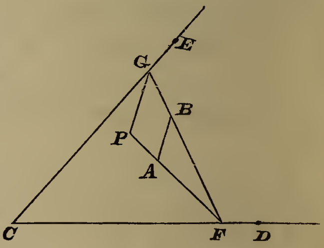
Hence, from (1),

$$n\,(kx-hy) = n'c\,(k-y) - n'\,(kx-h'y),$$
$$(n+n')\,kx - (nh+n'h'-n'c)\,y = n'ck,$$

the equation to the locus of *P*, which is therefore a straight line.

Leybourn's *Mathematical Repository*, *New Series*, vol. I. p. 45.

5. *A*, *B*, are two given points, *CD*, *CE*, two given straight lines. From any point *F* in *CD* is drawn the straight line *FBG*, cutting *CE* in *G*; from *G* is drawn *GP* parallel to *BA*, and meeting *FA* in *P*. To find the locus of *P*.



Let *CD*, *CE*, produced indefinitely, be taken as axes of x, y, respectively. Let a, b, be the coordinates of *A*; a', b', those of *B*. Also let $CF = m$.

Then the equation to *GBF* is

$$\frac{y}{b'} = \frac{x-m}{a'-m},$$

and thence, putting $x=0$, and $y = CG$, we have

$$CG = -\frac{b'm}{a'-m}.$$

Hence the equation to GP, which passes through G in a direction parallel to BA, is

$$y + \frac{b'm}{a' - m} = \frac{b' - b}{a' - a} x,$$

or $(a' - m)\{(a' - a)y + bx\} = b'\{x(a' - m) - m(a' - a)\}$...(1).

Again, the equation to FP, which passes through F and A, is

$$\frac{y}{b} = \frac{x - m}{a - m} \text{..........................(2).}$$

Eliminating y between (1) and (2), we have

$$\{b(a' - m) - b'(a - m)\}.\{x(a' - m) - m(a' - a)\} = 0,$$

and therefore $x(a' - m) - m(a' - a) = 0$,

and therefore, by (1), $(a' - a)y + bx = 0$,

the equation to the required locus, which is therefore a straight line passing through C.

Leybourn: *Mathematical Repository, New Series*, vol. I. p. 143.

6. The sides containing a given angle are in a given ratio and the vertex is fixed; supposing the extremity of one of the sides to move in a given straight line, to find the locus of the extremity of the other.

Let OA, OB, be the two sides; let $\angle AOB = \alpha$, $\angle AOx = \theta$, $\angle BOx = \theta'$, Ox being a fixed straight line; $OA = r$, $OB = r'$, $r' = nr$.

Then, since A moves in a straight line,

$$r \cos(\theta - \beta) = \delta,$$

where β and δ are constants.

Hence
$$r' \cos(\theta - \beta) = n\delta,$$
$$r' \cos(\theta' - \alpha - \beta) = n\delta,$$

the equation to the locus of B, which is therefore a straight line at a distance $n\delta$ from O, the inclination of this distance to Ox being $\alpha + \beta$.

7. From a fixed point A a straight line AB is drawn to a variable point B in a given straight line Ox, and upon AB is described an equilateral triangle ABP. To find the locus of P.

Let yAO, drawn through A at right angles to Ox, be taken as the axis of y, Ox being that of x, and let $AO = c$. Then the required locus will be a straight line represented by the equation

$$x\sqrt{3} - y = c.$$

8. To find the locus of a point, the distances of which from two lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1,$$

referred to rectangular coordinates, are always in the ratio of $n' : n$.

The required locus is a straight line defined by the equation

$$n.\frac{\frac{x}{a} + \frac{y}{b} - 1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{\frac{1}{2}}} = n'.\frac{\frac{x}{a'} + \frac{y}{b'} - 1}{\left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)^{\frac{1}{2}}}.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 122.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 190, *troisième édition.*

Garnier: *Géométrie Analytique*, p. 438, *deuxième édition.*

9. To find the locus of a point, the algebraical sum of the distances of which from the sides of a given polygon is constant.

If the equations to the sides of the polygon be

$$x \cos\alpha + y \sin\alpha = \delta, \quad x \cos\alpha' + y \sin\alpha' = \delta',$$
$$x \cos\alpha'' + y \sin\alpha'' = \delta'', \ldots\ldots,$$

the equation to the required locus will be

$$x(\cos\alpha + \cos\alpha' + \cos\alpha'' + \ldots) + y(\cos\beta + \cos\beta' + \cos\beta'' + \ldots)$$
$$= c + \delta + \delta' + \delta'' + \ldots\ldots,$$

where c is the constant quantity.

Timmermans: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 217.

10. Having given the positions of two points C, C', to find the locus of a point P under the condition that

$$(PC')^2 - (PC)^2 = \text{a constant quantity.}$$

Let a, b, be the coordinates of C; a', b', of C'; and x, y, of P. Then, the axes being supposed to be rectangular and c^2 denoting the value of the constant quantity, the required locus will be a straight line at right angles to CC', defined by the equation

$$(a - a').(2x - a - a') + (b - b').(2y - b - b') = c^2.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 121.

11. A parallel to the base of a triangle being drawn, and its points of intersection with the sides being connected with the extremities of the base, to find the locus of the intersection of the connecting lines.

The required locus is a right line bisecting the base and passing through the vertex.

Lamé: *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*, p. 47.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 122, *troisième édition.*

Lardner: *Algebraic Geometry*, vol. I. p. 28.

Garnier: *Géométrie Analytique*, p. 436, *deuxième édition.*

12. Ox, Oy, are two fixed indefinite straight lines; from a point P are drawn PA, PB, cutting Ox, Oy, respectively in

A, B: supposing the angles PAx, PBy, and the ratio of PA to PB to be given, to find the locus of P.

Let Ox, Oy, be taken as axes of x, y. Let $\angle PAx = \alpha$, $\angle PBy = \beta$, and let PA be to PB as m to n. Then the equation to the locus of P will be

$$mx \sin \alpha = ny \sin \beta.$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 249.

13. A straight line DCH, passing through two given points D, C, cuts a given straight line AB in H, in such a manner that

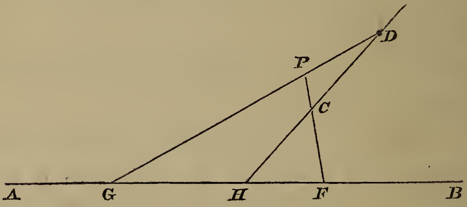
$$\frac{AH}{BH} = \frac{n}{n'},$$

n and n' being constants. In the line AB, are taken any two points F, G, such that

$$\frac{AF}{BG} = \frac{AH}{BH}.$$

To find the locus of P, the intersection of FC, GD, produced indefinitely.

Let HB, HCD, produced indefinitely, be taken as axes of x, y, respectively; let $HC = k$, $HD = k'$. Then the required locus will be a straight line, parallel to AB, the equation of which is



$$\left(\frac{n'}{k'} - \frac{n}{k}\right) y + n' - n = 0.$$

Leybourn: *Mathematical Repository, New Series*, vol. I. p. 45.

14. AB, AC, are two straight lines given in position; a straight line DE meets them in D, E, respectively, so that

$$AD + AE = \text{a constant length};$$

also DE is divided in a point P, so that DP bears a constant ratio to EP. To find the locus of P.

Taking AD, AE, produced indefinitely, as axes of x, y, respectively, let the constant value of $AD + AE$ be denoted by c, and let

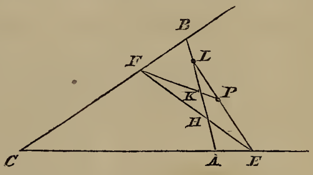
$$\alpha . DP = \beta . EP;$$

then the required locus will be a straight line defined by the equation

$$\frac{x}{\alpha} + \frac{y}{\beta} = \frac{c}{\alpha + \beta}.$$

Leybourn: *Mathematical Repository, New Series*, vol. II. p. 7.

15. ACB is a given triangle, H, K, L, given points in the side AB; through H is drawn any straight line EHF, cutting AC, BC, in E, F, respectively; through E and F are drawn right lines EL, FK, intersecting in P. To find the locus of P.



Let $HL = a$, $KB = b$, $HK = c$, $AL = d$, $AB = m$, $AC = n$; and let CE, produced indefinitely, and an indefinite line through C, parallel to AB, be taken respectively as axes of x and y. Then the locus of P will be a straight line defined by the equation

$$mcdx = (ab + cd).ny.$$

Leybourn: *Mathematical Repository, New Series*, vol. II. p. 89.

16. The loci of two points A, B, are two parallel straight lines. To find the locus of a third point P, such that PA, PB, are inclined at given angles to the loci of A, B, respectively, and bear to each other a constant ratio.

Let α, β, be the two given angles, and c the distance between the two parallel lines; also let

$$PA : PB :: m : n.$$

Then, the locus of A being taken as the axis of x, and that of y being at right angles to it, the equation to the required locus is

$$y = \frac{mc \sin \alpha}{m \sin \alpha + n \sin \beta}.$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 251.

SECTION V.

Transversals. Explicit Parameters.

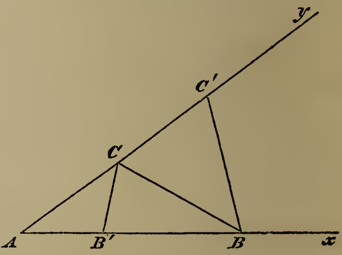
The word *transversal* is thus defined by Carnot, in his *Essai sur la Théorie des Transversales:*

"J'appelle *transversale* une ligne droite ou courbe qui traverse d'une manière quelconque un système d'autres lignes, soit droites, soit courbes; ou même un système de plans ou de surfaces courbes."

1. BAC is a triangle, and B', C', are points so taken in the sides AB, AC, respectively, or these sides produced, that

$$AB' + AC' = AB + AC.$$

To find the locus of the intersection of BC' and $B'C$, produced indefinitely.



Let AB, AC, produced indefinitely, be taken as axes of x, y, respectively.

Let $AB=a$, $AC=b$, $AB'=a'$, $AC'=b'$. Then the equation to $B'C$ will be

$$\frac{x}{a'} + \frac{y}{b} = 1 \text{..........................(1)},$$

and that to BC' will be

$$\frac{x}{a} + \frac{y}{b'} = 1 \text{..........................(2)}.$$

Also, by hypothesis, $a' + b' = a + b$ ....(3).

From (1) and (2), there is

$$x\frac{a-a'}{aa'} + y\frac{b'-b}{bb'} = 0,$$

and therefore, by (3),

$$\frac{x}{aa'} + \frac{y}{bb'} = 0 \text{ .... } (4).$$

But, from (1) and (2),

$$\frac{x}{aa'} + \frac{y}{ab} = \frac{1}{a}, \text{ and } \frac{y}{bb'} + \frac{x}{ab} = \frac{1}{b},$$

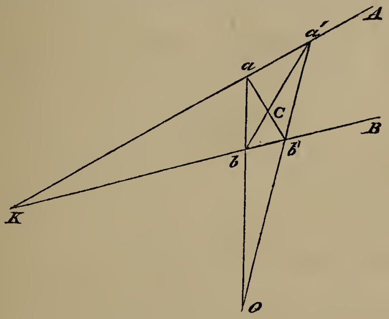
and therefore, by addition, attention being paid to (4),

$$x + y = a + b,$$

the equation to the required locus.

2. If A, B, O, be two fixed lines and a fixed point; then, drawing any two lines Oba, $Ob'a'$, through O, and joining the points a, b', and a', b, the locus of the point C, in which ab' and $a'b$ intersect, is a fixed line passing through the point of intersection of A and B.

Let the intersection K of A, B, be the origin of coordinates,



A and B being the axes of x and y. Let $Ka = a$, $Ka' = a'$, $Kb = b$, $Kb' = b'$; and let (λ, μ) be the coordinates of O.

Then the equation to ab' is

$$\frac{x}{a} + \frac{y}{b'} = 1,$$

and to $a'b$ is

$$\frac{x}{a'} + \frac{y}{b} = 1.$$

At their intersection C,

$$x\left(\frac{1}{a}-\frac{1}{a'}\right)+y\left(\frac{1}{b'}-\frac{1}{b}\right)=0 \dots\dots\dots\dots\dots (1).$$

But, since ab, $a'b'$, both pass through O,

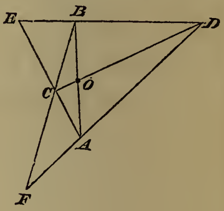
$$\frac{\lambda}{a}+\frac{\mu}{b}=1, \quad \frac{\lambda}{a'}+\frac{\mu}{b'}=1,$$

and therefore $$\lambda\left(\frac{1}{a}-\frac{1}{a'}\right)+\mu\left(\frac{1}{b}-\frac{1}{b'}\right)=0:$$

whence, by (1), $$\frac{x}{\lambda}+\frac{y}{\mu}=0,$$

the equation to the locus of C, which is therefore a straight line through K.

3. If two lines AB, CD, intersect in O, and if the lines AC, BD, meet, when produced, in E, and AD, BC, in F; to determine the condition that the line EF may be parallel to AB.



Let OA, OC, be taken as the axes of x, y, respectively. Let $OA = a$, $OB = b$, $OC = c$, $OD = d$. Then the equation to AC is

$$\frac{x}{a}+\frac{y}{c}=1;$$

and the equation to BD is

$$-\frac{x}{b}-\frac{y}{d}=1:$$

hence, at the point E, $$\left(\frac{a}{c}-\frac{b}{d}\right)y=a+b \dots\dots\dots\dots\dots\dots\dots (1).$$

Again, the equation to AD is

$$\frac{x}{a}-\frac{y}{d}=1,$$

and the equation to BC is

$$-\frac{x}{b}+\frac{y}{c}=1:$$

hence, at the point F,

$$\left(\frac{b}{c}-\frac{a}{d}\right) y = a + b \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In order that EF may be parallel to AB, the values of y in (1) and (2) must be equal: hence

$$\frac{b}{c}-\frac{a}{d}=\frac{a}{c}-\frac{b}{d},$$

$$a\left(\frac{1}{c}+\frac{1}{d}\right)=b\left(\frac{1}{c}+\frac{1}{d}\right),$$

$$a=b.$$

Thus the required condition is that AB be bisected in the point O.

F: *Cambridge Mathematical Journal*, vol. I. p. 87.

4. To prove that the perpendiculars from the three angles of a triangle upon the opposite sides all meet in one point.

Let (α, β), (α', β'), (α'', β''), be the three angular points. The equation to the side through (α', β'), (α'', β''), is

$$(\alpha''-\alpha')\, y-(\beta''-\beta')\, x=\alpha''\beta'-\alpha'\beta''.$$

The equation to the perpendicular upon it through the angle (α, β) is

$$(\alpha''-\alpha')\,(x-\alpha)+(\beta''-\beta')\,(y-\beta)=0,$$

or $$(\alpha''-\alpha')\, x+(\beta''-\beta')\, y=\alpha\,(\alpha''-\alpha')+\beta\,(\beta''-\beta')\ldots(1).$$

The equation to the perpendicular through (α', β') upon the opposite side is, by symmetry,

$$(\alpha-\alpha'')\, x+(\beta-\beta'')\, y=\alpha'\,(\alpha-\alpha'')+\beta'\,(\beta-\beta'')\ldots(2).$$

At the intersection of (1) and (2), we have, adding together their equations and changing signs,

$$(\alpha'-\alpha)\, x+(\beta'-\beta)\, y=\alpha''\,(\alpha'-\alpha)+\beta''\,(\beta'-\beta).$$

This equation coinciding in form with what we know by symmetry to be the equation to the third perpendicular, we see that the intersection of any two of the perpendiculars lies in the third.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 120, *troisième édition.*

5. To prove that lines drawn from the three angles of a triangle to bisect the opposite sides, all meet in a single point.

Let the coordinates of the three angles be (α, β), (α', β'), (α'', β''). The equation to the line through (α, β), and through the middle point between (α', β'), and (α'', β''), of which the coordinates are $\frac{1}{2}(\alpha' + \alpha'')$, $\frac{1}{2}(\beta' + \beta'')$, is

$$\frac{y - \beta}{\beta' + \beta'' - 2\beta} = \frac{x - \alpha}{\alpha' + \alpha'' - 2\alpha},$$

or $(\alpha' + \alpha'' - 2\alpha)\, y - (\beta' + \beta'' - 2\beta)\, x = \alpha'\beta - \alpha\beta' + \alpha''\beta - \alpha\beta'' \ldots (1)$.

The equation to the corresponding line through (α', β') is

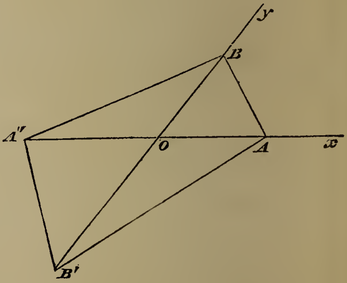
$(\alpha'' + \alpha - 2\alpha')\, y - (\beta'' + \beta - 2\beta')\, x = \alpha''\beta' - \alpha'\beta'' + \alpha\beta' - \alpha'\beta \ldots (2)$.

At the intersection of (1) and (2), we have, adding and changing signs,

$$(\alpha + \alpha' - 2\alpha'')\, y - (\beta + \beta' - 2\beta'')\, x = \alpha\beta'' - \alpha''\beta + \alpha'\beta'' - \alpha''\beta',$$

a result which shews that the lines through (α, β), (α', β'), cut each other in the line through (α'', β'').

6. In every quadrilateral, the three lines which join the middle points of the opposite sides and the middle points of the diagonals, pass through the same point.



Let $ABA'B'$ be any quadrilateral, the diagonals of which intersect in O.

Let OAx, OBy, be taken as the axes of coordinates.

Let $OA = a$, $OB = b$, $OA' = a'$, $OB' = b'$.

Then the coordinates of the middle points of AB, and $A'B'$, are respectively

$$\begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{2}a' \\ -\frac{1}{2}b' \end{pmatrix}.$$

Hence the equation to the line joining these middle points will be $(a' + a)\, y - (b' + b)\, x = \frac{1}{2}(a'b - ab') \ldots\ldots\ldots (1)$.

Again, the coordinates of the middle points of AB', $A'B$, are respectively

$$\begin{pmatrix} \tfrac{1}{2}a \\ -\tfrac{1}{2}b' \end{pmatrix} \text{ and } \begin{pmatrix} -\tfrac{1}{2}a' \\ \tfrac{1}{2}b \end{pmatrix}:$$

and therefore the equation to the line joining these points will be

$$(a' + a)\, y + (b' + b)\, x = \tfrac{1}{2}(ab - a'b') \quad \ldots\ldots\ldots(2).$$

Subtracting and adding the equations (1) and (2), we shall get, for the coordinates of the intersection of the lines they represent,

$$x = \tfrac{1}{4}(a - a'),$$
$$y = \tfrac{1}{4}(b - b').$$

Again, the intercepts on the axes of x and y of the line passing through the middle points of AA', BB', are

$$\tfrac{1}{2}(a - a'), \quad \tfrac{1}{2}(b - b').$$

These intercepts are accordingly twice the magnitudes of the coordinates of the intersection of (1) and (2); the intersection of (1) and (2) lies therefore in the line joining the middle points of the diagonals of the quadrilateral.

Gergonne: *Annales de Mathématiques*, tom. I. p. 353.

7. In every frustum of a triangle, with parallel bases, the intersection of the two diagonals is in a right line with the middle points of the bases of the frustum and the summit of the triangle.

Let the angle of the summit of the triangle be taken as the angle of the coordinates.

Then the equations to the bases of any frustum will be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{x}{\lambda a} + \frac{y}{\lambda b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations to the two diagonals will be

$$\frac{x}{a} + \frac{y}{\lambda b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{x}{\lambda a} + \frac{y}{b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

At the intersection of (3) and (4),

$$\frac{x}{a}\left(1-\frac{1}{\lambda}\right)-\frac{y}{b}\left(1-\frac{1}{\lambda}\right)=0,$$

or

$$\frac{x}{a}=\frac{y}{b} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (5),$$

which is the equation to a line which passes through the origin and through the intersection of every pair of diagonals.

Combining (1) and (5) we have

$$x=\tfrac{1}{2}a, \quad y=\tfrac{1}{2}b;$$

combining (2) and (5) we have

$$x=\tfrac{1}{2}\lambda a, \quad y=\tfrac{1}{2}\lambda b.$$

These results shew that the line (5) bisects the bases of every frustum.

Frégier: *Gergonne*, *Annales de Mathématiques*, tom. VII. p. 167.

8. Let AB, $A'B'$, $A''B''$, be any three parallel lines. Let C, C', C'', be the points of concourse respectively of $A'A''$, $B'B''$; of $A''A$, $B''B$; and of AA', BB'. Then the three points C, C', C'', will lie in a right line.

Take the axis of x parallel to the three lines AB, $A'B'$, $A''B''$. Then the equations to the extremities of these three lines will be,

$$A\ldots\begin{pmatrix}x=a\\y=b\end{pmatrix}, \qquad A'\ldots\begin{pmatrix}x=a'\\y=b'\end{pmatrix}, \qquad A''\ldots\begin{pmatrix}x=a''\\y=b''\end{pmatrix},$$

$$B\ldots\begin{pmatrix}x=a+c\\y=b\end{pmatrix}, \quad B'\ldots\begin{pmatrix}x=a'+c'\\y=b'\end{pmatrix}, \quad B''\ldots\begin{pmatrix}x=a''+c''\\y=b''\end{pmatrix}.$$

Hence we get for the equations of AA', BB',

$$AA' \ldots\ldots\; y-b=\frac{b-b'}{a-a'}(x-a),$$

$$BB' \ldots\ldots\; y-b=\frac{b-b'}{a-a'+c-c'}(x-a-c):$$

whence we have for the equations to the point C'', where these two lines meet,

$$C''\left\{\begin{aligned} x &= \frac{a'c - ac'}{c - c'} \\ y &= \frac{b'c - bc'}{c - c'} \end{aligned}\right\}.$$

By symmetry we have also,

$$C\left\{\begin{aligned} x &= \frac{a''c' - a'c''}{c' - c''} \\ y &= \frac{b''c' - b'c''}{c' - c''} \end{aligned}\right\}$$

and

$$C'\left\{\begin{aligned} x &= \frac{ac'' - a''c}{c'' - c} \\ y &= \frac{bc'' - b''c}{c'' - c} \end{aligned}\right\}$$

Let the axis of y be so chosen as to pass through C and C': we shall then have

$$\frac{a''c' - a'c''}{c' - c''} = 0, \qquad \frac{ac'' - a''c}{c'' - c} = 0,$$

or
$$a''c' = a'c'', \qquad ac'' = a''c,$$

whence also
$$a'c = ac',$$

and consequently
$$\frac{a'c - ac'}{c - c'} = 0:$$

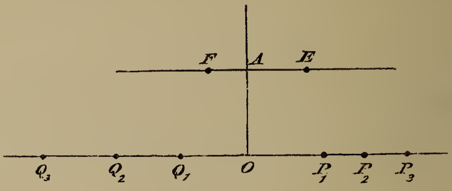
the point C'' will therefore lie also in the axis of y.

Durrande: *Gergonne, Annales de Mathématiques*, tom. VII. p. 183.

9. If, from any point O in a given line of indefinite length, equal distances OP_1, P_1P_2, P_2P_3, ... be measured off, and other equal distances OQ_1, Q_1Q_2, Q_2Q_3, ... in the opposite direction; and if P_1, P_2, P_3, ... be joined by straight lines with a point E, and Q_1, Q_2, Q_3, ... with a point F, E and F lying in a line parallel to the given line; then the intersections of lines of the former with lines of the latter group will all lie in

straight lines forming two groups of lines converging to two points in the line passing through E, F.

Let $OP_1P_2\dots$ be taken as the axis of x, and a line OA



through O, perpendicular to the two parallel lines, as the axis of y.

Let $OP_1 = a$, $OQ_1 = b$, $OA = c$, $AE = m$, $AF = n$.

Then, the intercepts of the line through E, P_r, are, as may be readily seen from similar triangles,

$$ra, \quad \frac{ra \, . \, c}{ra - m}:$$

hence the equation to EP_r is

$$\frac{x}{a} + \left(r - \frac{m}{a}\right)\frac{y}{c} = r \quad\dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Similarly, putting s for r, $-b$ for a, $-n$ for m, we have for the equation to FQ_s,

$$-\frac{x}{b} + \left(s - \frac{n}{b}\right)\frac{y}{c} = s \quad\dots\dots\dots\dots\dots\dots\dots\dots (2).$$

At the intersection of (1) and (2) we have, adding their equations,

$$x\left(\frac{1}{a} - \frac{1}{b}\right) + \left(r + s - \frac{m}{a} - \frac{n}{b}\right)\frac{y}{c} = r + s \quad\dots\dots\ (3),$$

and, subtracting,

$$x\left(\frac{1}{a} + \frac{1}{b}\right) + \left(r - s - \frac{m}{a} + \frac{n}{b}\right)\frac{y}{c} = r - s \quad\dots\dots\ (4).$$

The equations (3) and (4) shew that the intersections of the two groups of lines always take place in one of a single pair of lines so long as $r + s$ or $r - s$ is constant, there being as many such pairs as there are values of $r + s$ and $r - s$.

Suppose $y = c$: then, from (3),

$$x = \frac{\frac{m}{a} + \frac{n}{b}}{\frac{1}{a} - \frac{1}{b}},$$

and, from (4),

$$x = \frac{\frac{m}{a} - \frac{n}{b}}{\frac{1}{a} + \frac{1}{b}}.$$

These values of x, as not involving r or s, shew that the lines of intersection (3) and (4) all pass through two points in EF, at distances

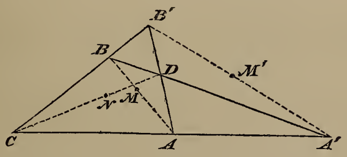
$$\frac{\frac{m}{a} + \frac{n}{b}}{\frac{1}{a} - \frac{1}{b}}, \quad \frac{\frac{m}{a} - \frac{n}{b}}{\frac{1}{a} + \frac{1}{b}},$$

from the point A.

F: *Cambridge Mathematical Journal*, vol. I. p. 88.

10. To prove that the middle points of the diagonals of the three simple quadrilaterals, forming a complete quadrilateral, lie all in one straight line.

The figure, of which A, B, C, D, A', B', are the angular points, is a complete quadrilateral, consisting of the three simple quadrilaterals,



$ACBDA$, of which the diagonals are AB, CD;

$A'DB'CA'$, of which the diagonals are CD, $A'B'$;

$AB'BA'A$, of which the diagonals are $A'B'$, AB.

Let $CA = a$, $CA' = a'$, $CB = b$, $CB' = b'$. Let M, M', be the middle points respectively of the diagonals AB, $A'B'$.

Through these two points draw a straight line cutting CD in N.

Let the lines CA', CB', be taken as axes of x, y, respectively. The equation to AB' is

$$\frac{x}{a}+\frac{y}{b'}=1;$$

and the equation to $A'B$ is

$$\frac{x}{a'}+\frac{y}{b}=1.$$

The equations to the point D are therefore

$$x=aa'.\frac{b-b'}{ab-a'b'},\quad y=bb'.\frac{a-a'}{ab-a'b'}.$$

The equation to CD will therefore be

$$aa'(b-b')\,y=bb'(a-a')\,x.$$

The equations to M are

$$x=\tfrac{1}{2}a,\quad y=\tfrac{1}{2}b;$$

and those to M' are

$$x=\tfrac{1}{2}a',\quad y=\tfrac{1}{2}b'.$$

Hence the equation to MM' is

$$2(a-a')\,y-2(b-b')\,x=ab'-a'b.$$

Combining this equation with that to CD, we shall have for the equations to N,

$$x=\tfrac{1}{2}aa'.\frac{b-b'}{ab-a'b'},\quad y=\tfrac{1}{2}bb'.\frac{a-a'}{ab-a'b'}.$$

The coordinates of N are therefore respectively half of those of D. The point N is accordingly the middle point of CD. Thus the middle points of the three diagonals are in a single straight line.

Rochat: *Gergonne*, *Annales de Mathématiques*, tom. I. p. 314.

11. Having given the position of a point relatively to a known angle and in the same plane with it, to find in this plane two other points such that if we draw through them, in an

arbitrary direction, two parallel straight lines cutting the two sides of the angle, the given point shall always lie in the direction of one of the diagonals of the trapezium intercepted between the parallels and the two sides of the given angle.

Let the two sides of the angle be taken as axes of coordinates. Then the equations to a pair of parallels will be, (x', y') and (x'', y'') being the two required points P', P'',

$$y - y' = a(x - x'),$$

and

$$y - y'' = a(x - x'').$$

The abscissa of the intersection of the former of these lines with the axis of x will be

$$\frac{ax' - y'}{a},$$

and the ordinate of the intersection of the latter with the axis of y will be

$$y'' - ax''.$$

Hence, (h, k) being the given point C, which by hypothesis lies in the line joining these two intersections, we have

$$\frac{ah}{ax' - y'} + \frac{k}{y'' - ax''} = 1,$$

or $x''.(x' - h).a^2 - \{y''(x' - h) + x''y' - kx'\}a + y'(y'' - k) = 0.$

Since a is indeterminate, this equation resolves itself into the three following:

$$x''(x' - h) = 0,$$
$$y'(y'' - k) = 0,$$
$$y''(x' - h) + x''y' - kx' = 0.$$

The problem is therefore indeterminate, since there are only three equations between the four coordinates x', y', x'', y''.

The first two equations can be satisfied by one of the four systems of values:

$$\begin{pmatrix} x'' = 0 \\ y' = 0 \end{pmatrix}, \quad \begin{pmatrix} x'' = 0 \\ y'' = k \end{pmatrix}, \quad \begin{pmatrix} x' = h \\ y' = 0 \end{pmatrix}, \quad \begin{pmatrix} x' = h \\ y'' = k \end{pmatrix}.$$

It is easily seen that of these systems only the first and the last can satisfy the third equation.

The first, which shews that the point P' is in the axis of x, and the point P'' in the axis of y, reduces the third equation to

$$x'(y''-k)-hy''=0;$$

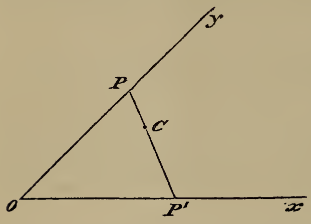
from which it appears that the two points P', P'', are in a right line with the point C. Thus the required points will be the intersections of the two sides of the given angle with any line whatever drawn through the given point.

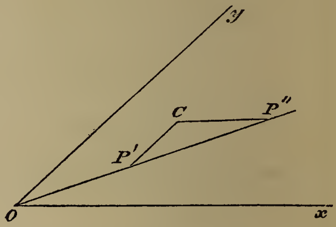
The last system, which indicates that the required points lie in lines drawn parallel to the two axes through the point C, reduces the third equation to

$$x''y'-kx'=0,$$

or
$$x''y'-y''x'=0,$$

which shews that the points P', P'', are in a right line with the origin. Thus we see that the points sought will be the intersections of a straight line, drawn arbitrarily through the summit of the given angle, with parallels to its two sides passing through the given point.





Schumacher: *Gergonne, Annales de Mathématiques*, tom. I. p. 193.
Garnier: *Géométrie Analytique*, p. 53.

12. The equations to two lines being

$$\frac{x}{a}+\frac{y}{b}=1, \quad \frac{x}{a'}+\frac{y}{b'}=1,$$

to find the equation to a line passing through their point of intersection and through the origin of coordinates, a, b, a', b', being subject to the relation

$$a+b=a'+b'.$$

The required equation is

$$\frac{x}{aa'} = \frac{y}{bb'}.$$

13. To find the equations to the diagonals of the parallelogram, the sides of which are defined by the equations

$$x = a, \quad x = a', \quad y = b, \quad y = b',$$

and to determine the coordinates of the point of their intersection.

The equations to the two diagonals are

$$(a' - a)\,y - a'b = (b' - b)\,x - b'a,$$

and

$$(a - a')\,y - ab = (b' - b)\,x - a'b';$$

and the coordinates of their intersection

$$\tfrac{1}{2}(a + a'), \quad \tfrac{1}{2}(b + b').$$

14. To find the condition that the three lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1, \quad \frac{x}{a''} + \frac{y}{b''} = 1,$$

may all pass through a single point.

The condition is expressed by the following equation,

$$\frac{1}{a'b''} - \frac{1}{a''b'} + \frac{1}{a''b} - \frac{1}{ab''} + \frac{1}{ab'} - \frac{1}{a'b} = 0.$$

Lamé: *Gergonne*, *Annales de Mathématiques*, tom. VII. p. 230.

15. If p_1, p_2, p_3, denote the perpendicular distances of any point in a side of any quadrilateral from the three other sides; and a, b, assigned lengths; to prove that the equation

$$\frac{p_1}{a}\,x + \frac{p_2}{b}\,y = p_3,$$

will represent one of a family of lines which all pass through a single point.

16. To find the equation to a straight line passing through the intersection of the two lines

$$x\cos\varepsilon + y\sin\varepsilon = \delta, \qquad x\cos\varepsilon' + y\sin\varepsilon' = \delta',$$

and cutting at right angles the line

$$x\cos\varepsilon'' + y\sin\varepsilon'' = \delta''.$$

The required equation is

$$\frac{x\cos\varepsilon + y\sin\varepsilon - \delta}{\cos(\varepsilon'' - \varepsilon)} = \frac{x\cos\varepsilon' + y\sin\varepsilon' - \delta'}{\cos(\varepsilon'' - \varepsilon')}.$$

O'Brien: *Plane Coordinate Geometry*, p. 31.

17. If, on the three sides of a triangle, taken in turn as diagonals, be constructed parallelograms, the sides of which are parallel to two given straight lines, to prove that the other three diagonals of these parallelograms will pass through a single point.

Sturm, Vecten, Querret: *Gergonne*, *Annales de Mathématiques*, tom. XV. p. 103.

18. If, through the extremities of the base AB of any triangle ABC, be drawn straight lines AP, BQ, of arbitrary length, and respectively parallel to the sides BC, AC; and if, through the points P and Q, be drawn, parallel to BQ, AP, respectively, straight lines meeting in a point D; to prove that the three lines AQ, BP, DC, will pass through a single point.

Gergonne: *Annales de Mathématiques*, tom. XV. p. 88.

19. If any two straight lines be drawn, one of which bisects the other, and the straight lines joining their extremities be produced to intersect, to prove that the line joining the two points of intersection shall be parallel to the line bisected.

20. If $OAA'A''$, $OBB'B''$, be two straight lines, harmonically* divided, the former in the points A, A', and the latter in the

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* A straight line $ABCD$ is said to be harmonically divided in the points B, C, if the whole AD : either extreme AB :: the other extreme CD : the mean BC.

points B, B', to prove that the straight lines AB, $A'B'$, $A''B''$, will, produced indefinitely if necessary, either pass through a single point, or be parallel to each other.

De La Hire: *Sectiones Conicæ*, lib. I. prop. 18.

21. From three points A, B, D, in a straight line $ABCD$, straight lines are drawn through a single point P; through C is drawn a straight line ECF, parallel to AP, cutting PB produced in E, and PD in F. To prove that

$$AD.BC : AB.CD :: CE : CF.$$

De La Hire: *Sectiones Conicæ*, Appendix, prop. 3.

22. From an angle C of a parallelogram $ABCD$, a perpendicular CE is drawn to the diagonal BD; to prove that perpendiculars drawn to AB, AD, at the points B, D, will intersect each other in the line CE.

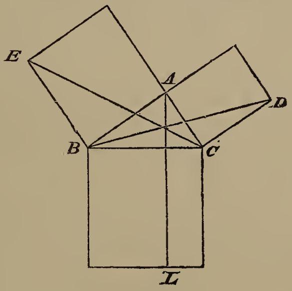
Leybourn: *Mathematical Repository*, *New Series*, vol. III. p. 179.

23. APB, CQD, are two parallel straight lines. Also

$$AP : PB :: DQ : QC;$$

to prove that the right lines PQ, AD, BC, meet in a single point.

24. Euclid I. 47. In the figure AL is perpendicular to BC;



to prove algebraically that DB, EC, intersect AL in the same point.

25. If through the three angular points of any given triangle, lines be drawn parallel to the opposite sides, to prove that the straight lines joining their points of intersection and the opposite angles of the original triangle will intersect each other in a single point.

SECTION VI.

Transversals. Implicit Parameters.

1. To prove that the three lines bisecting the angles of a triangle all pass through the same point.

Let the equations to the three sides be

$$u = 0, \quad v = 0, \quad w = 0,$$

where u, v, w, are of the form

$$x \cos \varepsilon + y \sin \varepsilon - \delta.$$

Then, the origin being anywhere within the area of the triangle, the equations to the bisecting lines will be

$$u - v = 0, \quad v - w = 0, \quad w - u = 0;$$

these equations shew that the bisecting lines pass through a single point.*

2. If lines be drawn, bisecting the angles of a triangle and the exterior angles formed by producing the sides, these lines will intersect in only four points besides the angles of the triangle.

Taking $x \cos \varepsilon + y \sin \varepsilon - \delta = 0$ as the general type of the equation to a straight line, we may represent the equations to the sides of the triangle by

$$u = 0, \quad v = 0, \quad w = 0,$$

and the equations to the six bisecting lines by the equations

$$v + w = 0 \ldots (1), \quad w + u = 0 \ldots (2), \quad u + v = 0 \ldots (3),$$
$$v - w = 0 \ldots (1)', \quad w - u = 0 \ldots (2)', \quad u - v = 0 \ldots (3)'.$$

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* See problem (5) of this section.

The pairs of lines (1) and (1)′; (2) and (2)′; (3) and (3)′; meet respectively at the angles of the triangle. Also the triple systems of lines

$$\{(1), (2), (3)'\},\ \{(2), (3), (1)'\},\ \{(3), (1), (2)'\},\ \{(1)', (2)', (3)'\},$$

meet each in a point.

Now, generally, the number of intersections of six lines is equal to 15: hence, in the present problem, deducting from this number the number 3, for intersections at the angles of the triangle, and the number 2, for each of the triple systems which pass through only one point instead of three, we have, for the greatest number of intersections,

$$15 - 3 - 8 = 4.$$

3. From the angles A, B, C, of any triangle are drawn three straight lines AA', BB', CC', bisecting the angles of the triangle: through A, B, C, are also drawn three straight lines, respectively perpendicular to AA', BB', CC', to meet the sides BC, CA, AB, produced, respectively in G, H, K. To prove that G, H, K, lie in a right line.

Taking $x \cos \varepsilon + y \sin \varepsilon = \delta$ as the general type of the equation to a straight line, the equations to the several lines of the figure will be

$BC \ldots\ldots\ldots u = 0$, $CA \ldots\ldots\ldots v = 0$, $AB \ldots\ldots\ldots w = 0$,
$AA' \ldots v - w = 0$, $BB' \ldots w - u = 0$, $CC' \ldots. u - v = 0$,
$AG \ldots v + w = 0$, $BH \ldots. w + u = 0$, $CK \ldots.. u + v = 0$.

From the equations to BC and AG, it is evident that G will lie in a line, of which the equation is

$$u + v + w = 0.$$

Symmetry shews that this line must contain also H and K.

Vecten: *Gergonne*, *Annales de Mathématiques*, tom. x. p. 202.

4. To find the equation to a straight line which passes through a proposed point and through the intersection of the two lines

$$u = 0, \quad v = 0.$$

The general equation to a line passing through the intersection of these two lines must be

$$u = \lambda v \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

λ being an arbitrary constant. This is evident when we consider that the values of x and y, which satisfy the equations $u = 0$, $v = 0$, must also satisfy the required equation; or, which is the same thing, that whenever $u = 0$ and $v = 0$, simultaneously, the required equation must be identically satisfied.

Let u_1, v_1, be the values of u, v, when the coordinates of the proposed point are substituted for x, y. Then, since the equation (1) must be satisfied by these values of x and y, we have

$$u_1 = \lambda v_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

From (1) and (2) we have

$$\frac{u}{u_1} = \frac{v}{v_1},$$

the equation to the required line.

Suppose for illustration the two proposed lines to be

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \frac{x}{a'} + \frac{y}{y'} = 1,$$

and the proposed point to be α, β. Then the required line will have for its equation

$$\frac{\dfrac{x}{a} + \dfrac{y}{b} - 1}{\dfrac{\alpha}{a} + \dfrac{\beta}{b} - 1} = \frac{\dfrac{x}{a'} + \dfrac{y}{b'} - 1}{\dfrac{\alpha}{a'} + \dfrac{\beta}{b'} - 1}.$$

5. To prove that any three lines, which intersect in one point, may be represented by the three equations

$$v - w = 0, \quad w - u = 0, \quad u - v = 0.$$

Let the equations to any two of the lines be

$$p = 0, \quad q = 0:$$

that to the third will be

$$p + \lambda q = 0.$$

Hence any three lines whatever, which cut each other in one point, may be represented by the equations

$$p = 0, \quad \lambda q = 0, \quad -p - \lambda q = 0;$$

or, putting $p = v - w$, $\lambda q = w - u$,

by the three equations

$$v - w = 0, \quad w - u = 0, \quad u - v = 0.$$

6. Three straight lines A, B, C, pass through a single point; X, Y, Z, are three other straight lines such that Y, Z, intersect in A; Z, X, in B; and X, Y, in C. To prove that the lines A, B, C, X, Y, Z, may be respectively represented by the correlative system of equations

$$v - w = 0, \quad w - u = 0, \quad u - v = 0,$$
$$v + w = \lambda, \quad w + u = \lambda, \quad u + v = \lambda.$$

The first three equations, as we know by problem (5), will always represent A, B, C.

Again, since Y, Z, intersect in A, the equations to Y, Z, must be of the forms

$$p = 0, \quad ap + v - w = 0;$$

or, putting $ap = \lambda - u - v$, of the forms

$$u + v = \lambda,$$
$$w + u = \lambda.$$

Again, it is obvious that a line of which the equation is

$$v + w = \lambda,$$

intersects Y in B, and Z in C: this line must therefore necessarily be X.

Hence the system of equations is

A $v - w = 0$	B $w - u = 0$	C $u - v = 0$
X $v + w = \lambda$	Y $w + u = \lambda$	Z $u + v = \lambda$

7. HKL, PQR, are two triangles; to prove that if the straight lines HP, KQ, LR, meet in one point O, the intersections of KL, QR; LH. RP; HK, PQ; lie in a straight line.

Since OPH, OQK, ORL, all meet in a point, their equations may be expressed respectively in the forms

$$v - w = 0, \quad w - u = 0, \quad u - v = 0.$$

Again, since KL, LH; LH, HK; HK, KL; intersect respectively in ORL, OPH, OQK, the equations to KL, LH, HK, are

$$\lambda = v + w, \quad \lambda = w + u, \quad \lambda = u + v.$$

For a like reason, the equations to QR, RP, PQ, are respectively

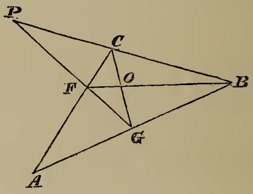
$$\lambda' = v + w, \quad \lambda' = w + u, \quad \lambda' = u + v.$$

Hence KL, QR; LH, RP; HK, PQ; all intersect in a straight line

$$\lambda = \lambda'.$$

Gergonne: *Annales de Mathématiques*, tom. VII. p. 187.
Vincent: *Cours de Géométrie*, p. 210.
Greathead: *Cambridge Mathematical Journal*, vol. I. p. 171.
Frost: *ib.* vol. IV. p. 113.

8. From the angles A, B, C, of a triangle, lines are drawn, through a point O, to meet the opposite sides in E, F, G, respectively: FG, GE, EF, are produced to meet BC, CA, AB, respectively, in P, Q, R. To prove that P, Q, R, lie in one line.



Let the equations to BC, CA, AB, be

$$u = 0, \quad v = 0, \quad w = 0.$$

The equations to CG, BF, are respectively

$$\frac{u}{u_1} = \frac{v}{v_1},$$

and

$$\frac{u}{u_1} = \frac{w}{w_1},$$

where u_1, v_1, w_1, are the values of u, v, w, at the point O.

The equations to the points G, F, are therefore, respectively,

$$\left\{\begin{array}{l} \dfrac{u}{u_1} = \dfrac{v}{v_1} \\ \dfrac{w}{w_1} = 0 \end{array}\right\}$$

and

$$\left\{\begin{array}{l} \dfrac{u}{u_1} = \dfrac{w}{w_1} \\ \dfrac{v}{v_1} = 0 \end{array}\right\}.$$

Hence the line GFP is defined by the equation

$$\frac{v}{v_1} + \frac{w}{w_1} = \frac{u}{u_1}.$$

Hence the equations to the point P are

$$\left\{\begin{array}{l} \dfrac{v}{v_1} + \dfrac{w}{w_1} = 0 \\ \dfrac{u}{u_1} = 0 \end{array}\right\}.$$

P therefore lies in the line

$$\frac{u}{u_1} + \frac{v}{v_1} + \frac{w}{w_1} = 0.$$

Symmetry shews the same to be true of Q and R. Hence P, Q, R, lie in one line defined by this last equation.

We subjoin also another solution of this problem.

Since AE, BF, CG, meet in a point, their equations may be written respectively in the forms

AE $v - w = 0$......................(1),

BF $w - u = 0$......................(2),

CG $u - v = 0$......................(3).

Since AB, AC, intersect in (1); BC, BA, in (2); CA, CB, in (3); the equations to CA, AB, BC, will be respectively

CA $w + u = 2\lambda$.....................(4),

AB $u + v = 2\lambda$.....................(5),

BC $v + w = 2\lambda$.....................(6).

Since FG passes through the intersections of CA, BF, and of AB, CG; GE, EF, passing through analogous points; the equations to FG, GE, EF, will be respectively

$$FG \quad u = \lambda \dotfill (7),$$
$$GE \quad v = \lambda \dotfill (8),$$
$$EF \quad w = \lambda \dotfill (9).$$

Hence P, Q, R, the intersections of (6), (7); (4), (8); (5), (9); all lie in a line

$$u + v + w = 3\lambda.$$

F: *Cambridge Mathematical Journal*, vol. I. p. 87.

9. To prove that, the circumstances of the preceding problem remaining unchanged, the lines joining B, Q; C, R; A, E; meet in one point, as also those joining C, R; A, P; B, F; and those joining A, P; B, Q; C, G.

The equations to GF, BC, AB, CA, being

$$GF \qquad u = \lambda,$$
$$BC \quad v + w = 2\lambda,$$
$$AB \quad u + v = 2\lambda,$$
$$CA \quad w + u = 2\lambda,$$

it is plain that the line denoted by the equation

$$2u + v + w = 4\lambda$$

passes through the intersection of GF, BC; and of AB, CA; the line is therefore the line joining A, P.

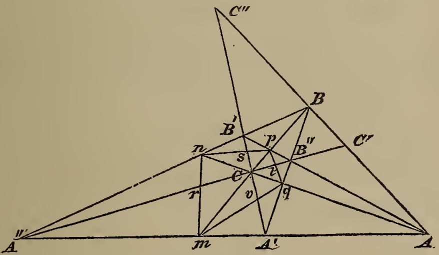
The equations to AP, BQ, CG, are therefore, regard being paid to symmetry,

$$AP \quad 2u + v + w = 4\lambda,$$
$$BQ \quad 2v + w + u = 4\lambda,$$
$$CG \qquad u - v = 0.$$

Since any one of these equations results from the other two, these three lines must all pass through one point. The same thing must, by virtue of symmetry, hold good respecting the intersections of the similarly related lines.

10. Let there be any complete quadrilateral; let its sides be produced indefinitely; let its three diagonals be also indefinitely produced: the diagonals will intersect, two and two, in three points. Through each of these points let there be drawn straight lines to the two extremities of the diagonal in which it does not lie; we shall thus have six straight lines, each of which will determine two points on two sides of the quadrilateral; so that we shall have in all twelve of these points, distributed, three and three, on the four sides of this quadrilateral. It is required to prove that these twelve points will lie, two and two, on twelve new straight lines, meeting, four and four, in the three points of intersection of the diagonals of the proposed quadrilateral.

Let $AA'A''B'BB''$ be the proposed quadrilateral, of which the diagonals are AB, $A'B'$, $A''B''$, intersecting, AB and $A'B'$ in C'',



AB and $A''B''$ in C', $A'B'$ and $A''B''$ in C. Let the point C be joined to the points A, B, by two straight lines, of which the former cuts the sides BA'', BA', in n, q, and the latter the sides AA'', AB', in m, p. Since the construction would evidently be the same for the point C', relatively to the diagonal $A'B'$, and for the point C'', relatively to the diagonal $A''B''$; it will be sufficient to prove, first, that the lines np, mq, meet in the point C', and, secondly, that the lines mn, qp, meet in the point C''.

Since the three lines AB', $A'B$, $A''C'$, meet in one point, we may take for their equations

$$AB' \ldots\ldots\ldots\ldots u - v = 0,$$
$$A'B \ldots\ldots\ldots\ldots v - w = 0,$$
$$A''C' \ldots\ldots\ldots\ldots w - u = 0.$$

Again, since $A''B$, AA'', AB, intersect, two and two, in AB', $A'B$, $A''C'$, their equations will be

$$A''B \ldots\ldots\ldots\ldots u + v = 2\lambda,$$

$$AA'' \ldots\ldots\ldots\ldots v + w = 2\lambda,$$

$$AB \ldots\ldots\ldots\ldots w + u = 2\lambda.$$

Since $A'B'$ passes through the intersection of AA'', $A'B$, and of $A''B$, AB', its equation will be

$$A'B' \ldots\ldots\ldots\ldots v = \lambda.$$

Since An passes through the intersection of AB, AA'', and of $A'B'$, $A''C'$, its equation will be

$$An \ldots\ldots\ldots\ldots - u + 2v + w = 2\lambda.$$

Since Bm passes through the intersection of $A'B'$, $A''C'$, and of AB, $A''B$, its equation will be

$$Bm \ldots\ldots\ldots\ldots u + 2v - w = 2\lambda.$$

Since mn passes through the intersection of Bm, AA'', and of An, $A''B$, its equation will be

$$mn \ldots\ldots\ldots\ldots u + 4v + w = 6\lambda.$$

Since pq passes through the intersection of AB', Bm, and of $A'B$, An, its equation will be

$$pq \ldots\ldots\ldots\ldots - u + 4v - w = 2\lambda.$$

At the intersection of mn, pq, we have, from their equations,

$$\begin{Bmatrix} v = \lambda \\ w + u = 2\lambda \end{Bmatrix};$$

their intersection therefore coincides with C'', the intersection of AB, $A'B'$.

Again, since np passes through the intersection of $A''B$, An, and of AB', Bm, its equation will be

$$np \ldots\ldots\ldots\ldots 3u - w = 2\lambda.$$

Since mq passes through the intersection of Bm, AA'', and of $A'B$, An, its equation will be

$$mq \ldots\ldots\ldots\ldots - u + 3w = 2\lambda.$$

At the intersection of *np* and *mq*, we have, from their equations,

$$\left\{\begin{array}{l} w - u = 0 \\ w + u = 2\lambda \end{array}\right\},$$

their intersection therefore coincides with C', the intersection of AB, $A''C'$.

COR. If *r*, *t*, be the intersections of *mn*, *pq*, with $A''B''$, and *s*, *v*, those of *np*, *mq*, with $A'B'$; *rs*, *vt*, will intersect in *B*, and *rv*, *st*, in *A*.

Availing ourselves of the preceding equations, we shall easily obtain for the equations to *rs*, *vt*, *rv*, *st*,

$$\begin{array}{ll} rs \ldots\ldots\ldots\ldots & 3u + 4v - w = 6\lambda, \\ vt \ldots\ldots\ldots\ldots & u + 4v - 3w = 2\lambda, \\ rv \ldots\ldots\ldots\ldots & -u + 4v + 3w = 6\lambda, \\ st \ldots\ldots\ldots\ldots & -3u + 4v + w = 2\lambda. \end{array}$$

From these equations the truth of the proposition asserted in the corollary is easily seen.

The truth of the corollary is also evident from the consideration that the four points *r*, *s*, *t*, *v*, have the same relation to the quadrilateral *mnpq*, which the four points *m*, *n*, *p*, *q*, have to the quadrilateral $A''A'B''B'$.

Legrand, Rochat, Penjon: *Gergonne*, *Annales de Mathématiques*, tom. II. p. 369.

The reader is referred also to a memoir by Vecten, in the 15th volume of the *Annales*, p. 146, where he will find other properties of the complete quadrilateral.

11. Let AB, $A'B'$, $A''B''$, be any three parallel lines. Let C, C', C'', be the points of concourse respectively of $A'A''$, $B'B''$, of $A''A$, $B''B$, and of AA', BB'. Then the three points C, C', C'', will lie in a right line.

The lines AB, $A'B'$, $A''B''$, $A'A''$, $B'B''$, AA', BB', may be represented by the system of equations

$$AB \ldots\ldots\ldots u = c, \quad A'B' \ldots\ldots\ldots u = c', \quad A''B'' \ldots\ldots\ldots u = c'';$$

$$A'A'' \ldots\ldots\ldots v = 0, \quad B'B'' \ldots\ldots\ldots w = 0;$$

$$AA' \ldots\ldots\ldots u - v = c', \quad BB' \ldots\ldots\ldots u - w = c''.$$

The equations to the points A, A'', will then be

$$A \ldots\ldots\ldots \left\{\begin{matrix} u = c \\ v = c - c' \end{matrix}\right\}, \quad A'' \ldots\ldots\ldots \left\{\begin{matrix} u = c'' \\ v = 0 \end{matrix}\right\}.$$

Hence the equation to the line $A''A$ will be

$$A''A \ldots\ldots\ldots (c' - c)\, u + (c - c'')\, v = c''\,(c' - c).$$

Again, the equations to the points B, B'', will be

$$B \ldots\ldots\ldots \left\{\begin{matrix} u = c \\ w = c - c' \end{matrix}\right\}, \quad B'' \ldots\ldots\ldots \left\{\begin{matrix} u = c'' \\ w = 0 \end{matrix}\right\}.$$

The equation to the line $B''B$ will therefore be

$$B''B \ldots\ldots\ldots (c' - c)\, u + (c - c'')\, w = c''\,(c' - c).$$

From the above equations it is plain that the intersections of $A'A''$, $B'B''$; of $A''A$, $B''B$; and of AA', BB'; will all three lie in a line of which the equation is

$$v = w.$$

Durrande: *Gergonne*, *Annales de Mathématiques*, tom VII. p. 183.

12. Let ASB, $A'S'B'$, $A''S''B''$, be three angles of which the sides are respectively parallel. Let $S'A'$, $S''B''$, meet in M, and $S'B'$, $S''A''$, in N: let $S''A''$, SB, meet in M', and $S''B''$, SA, in N': let SA, $S'B'$, meet in M'', and SB, $S'A'$, in N''. Then the three lines MN, $M'N'$, $M''N''$, will meet in one point.

Let the equations to the sides of the three angles be

$$AS \ldots\ldots\ldots\ldots u = \alpha, \qquad BS \ldots\ldots\ldots\ldots v = \beta;$$
$$A'S' \ldots\ldots\ldots u = \alpha', \qquad B'S' \ldots\ldots\ldots v = \beta';$$
$$A''S'' \ldots\ldots\ldots u = \alpha'', \qquad B''S'' \ldots\ldots\ldots v = \beta''.$$

Then the equations to the points M, N, will be

$$M \ldots\ldots\ldots \left\{\begin{matrix} u = \alpha' \\ v = \beta'' \end{matrix}\right\}, \quad N \ldots\ldots\ldots \left\{\begin{matrix} u = \alpha'' \\ v = \beta' \end{matrix}\right\}.$$

The equation to MN will be, therefore,

$$MN \ldots\ldots\ldots (\beta' - \beta'')\, u + (\alpha' - \alpha'')\, v = \alpha'\beta' - \alpha''\beta''.$$

By similarity, for the equations to $M'N'$, $M''N''$, we have

$$M'N'\ldots\ldots\ldots(\beta''-\beta)\,u+(\alpha''-\alpha)\,v=\alpha''\beta''-\alpha\beta,$$

$$M''N''\ldots\ldots\ldots(\beta-\beta')\,u+(\alpha-\alpha')\,v=\alpha\beta-\alpha'\beta'.$$

Adding together these last three equations, we obtain an identical equation; the three lines MN, $M'N'$, $M''N''$, therefore all meet in one point.

Durrande: *Gergonne*, *Annales de Mathématiques*, tom. VII. p. 183.

13. If three straight lines, drawn from the summits of a triangle, meet in one point, their respective parallels, drawn from the middle points of the opposite sides, also meet in one point.

The equations to the three former lines may be written in the forms

$$v-w=\mu-\nu\ldots\ldots\ldots\ldots OA,$$

$$w-u=\nu-\lambda\ldots\ldots\ldots\ldots OB,$$

$$u-v=\lambda-\mu\ldots\ldots\ldots\ldots OC;$$

and the equations to the three latter, being respectively parallel to the three former, in the forms

$$v-w=l\ldots(4),\qquad w-u=m\ldots(5),\qquad u-v=n\ldots\ldots(6).$$

In these six equations u, v, w, are supposed to denote linear functions of x and y without constant terms; l, m, n, λ, μ, ν, being all constants.

Let $v_{,}$, $w_{,}$, be the values of v, w, at the middle point of BC; $w_{,,}$, $u_{,,}$, those of w, u, at the middle point of CA; and $u_{,,,}$, $v_{,,,}$, those of u, v, at the middle point of AB.

Also let (u', v', w'), (u'', v'', w''), (u''', v''', w'''), be the values of (u, v, w), at the points A, B, C, respectively.

Then it is plain, from the geometrical conditions, that

$$2v_{,}=v''+v''',\qquad 2w_{,}=w''+w''';$$

$$2w_{,,}=w'''+w',\qquad 2u_{,,}=u'''+u';$$

$$2u_{,,,}=u'+u'',\qquad 2v_{,,,}=v'+v''.$$

But, from (4), (5), (6), we have

$$2\,(l+m+n)=2\,(v_{,}-w_{,})+2\,(w_{,,}-u_{,,})+2\,(u_{,,,}-v_{,,,});$$

hence
$$\begin{aligned}2(l+m+n) &= v'' - w'' + v''' - w''' \\ &\quad + w''' - u''' + w' - u' \\ &\quad + u' - v' + u'' - v'' \\ &= v''' - u''' + w' - v' + u'' - w''.\end{aligned}$$

But, from the equations to OA, OB, OC,

$$v' - w' = \mu - \nu, \quad w'' - u'' = \nu - \lambda, \quad u''' - v''' = \lambda - \mu:$$

hence
$$2(l+m+n) = \mu - \lambda + \nu - \mu + \lambda - \nu,$$

and therefore
$$l+m+n=0.$$

This result shews that the lines (4), (5), (6), pass through one point (O').

COR. It is easily seen that the equation to the line joining O, O', if we put $l = \mu' - \nu'$, $m = \nu' - \lambda'$, $n = \lambda' - \mu'$, is

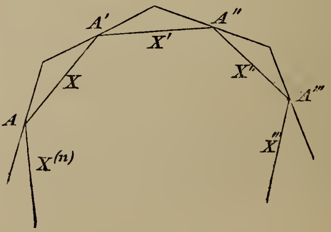
$$\begin{aligned}&(\lambda' - \lambda)(\mu - \nu) + (\mu' - \mu)(\nu - \lambda) + (\nu' - \nu)(\lambda - \mu) \\ &= (\lambda' - \lambda)(v - w) + (\mu' - \mu)(w - u) + (\nu' - \nu)(u - v) \\ &= (\lambda' - \lambda)(\mu' - \nu') + (\mu' - \mu)(\nu' - \lambda') + (\nu' - \nu)(\lambda' - \mu').\end{aligned}$$

Fregier: *Gergonne, Annales de Mathématiques*, tom. VII. p. 170.

14. To prove that the sides A, A', A'', A''', ... of any polygon, and the sides X, X', X'', X''', ... of any polygon inscribed within it, may be represented by the system of equations

$$\left|\begin{matrix}A\\ u=0\end{matrix}\right| \left|\begin{matrix}A'\\ u'=0\end{matrix}\right| \left|\begin{matrix}A''\\ u''=0\end{matrix}\right| \left|\begin{matrix}A'''\\ u'''=0\end{matrix}\right| \ldots\ldots \left|\begin{matrix}A^{(n)}\\ u^{(n)}=0\end{matrix}\right|;$$

$$\begin{array}{l}X\ldots\ldots\ldots au + u' + u'' + u''' + \ldots\ldots + u^{(n)} = 0,\\ X'\ldots\ldots\ldots au + au' + u'' + u''' + \ldots\ldots + u^{(n)} = 0,\\ X''\ldots\ldots\ldots au + au' + au'' + u''' + \ldots\ldots + u^{(n)} = 0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ X^{(n)}\ldots\ldots au + au' + au'' + au''' + \ldots\ldots + au^{(n)} = 0.\end{array}$$



Let the sides A, A', A'', ... of the polygon be represented by the equations

$$A = 0, \quad A' = 0, \quad A'' = 0, \ldots\ldots,$$

and the sides X, X', X'', ... by the equations

$$X = 0, \quad X' = 0, \quad X'' = 0, \ldots\ldots.$$

Since A, X, $X^{(n)}$, pass through a single point,

$$X^{(n)} = \lambda A + \mu X,$$

λ and μ being constants. In like manner

$$\begin{aligned} X &= \lambda_1 A' + \mu_1 X', \\ X' &= \lambda_2 A'' + \mu_2 X'', \\ X'' &= \lambda_3 A''' + \mu_3 X''', \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X^{(n-1)} &= \lambda_n A^{(n)} + \mu_n X^{(n)}. \end{aligned}$$

Multiply these $n + 1$ equations in order by

$$1, \quad \mu, \quad \mu\mu_1, \quad \mu\mu_1\mu_2, \ldots\ldots\ldots \mu\mu_1\mu_2, \ldots\ldots\ldots \mu_{n-1};$$

then, representing

$$X^{(n)}, \quad \mu X, \quad \mu\mu_1 X', \ldots\ldots\ldots \mu\mu_1\mu_2 \ldots\ldots\ldots \mu_{n-1} X^{(n-1)},$$

by $$Y^{(n)}, \quad Y, \quad Y', \ldots\ldots\ldots Y^{(n-1)},$$

and $$\lambda A, \quad \mu\lambda_1 A', \quad \mu\mu_1\lambda_2 A'', \ldots\ldots\ldots \mu\mu_1\mu_2. \ldots\ldots\ldots \mu_{n-1}\lambda_n A^{(n)},$$

by $$u, \quad u', \quad u'', \ldots\ldots\ldots u^{(n)},$$

we have, putting α for $\mu\mu_1\mu_2 \ldots\ldots \mu_{n-1}$,

$$\begin{aligned} Y^{(n)} &= u + Y, \\ Y &= u' + Y', \\ Y' &= u'' + Y'', \\ Y'' &= u''' + Y''', \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Y^{(n-1)} &= u^{(n)} + \alpha Y^{(n)}. \end{aligned}$$

From these equations we have

$$\begin{aligned}
(1-\alpha)Y^{(n)} &= u + u' + u'' + \ldots\ldots + u^{(n)},\\
(1-\alpha)Y &= \alpha u + u' + u'' + \ldots\ldots + u^{(n)},\\
(1-\alpha)Y' &= \alpha u + \alpha u' + u'' + \ldots\ldots + u^{(n)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(1-\alpha)Y^{(n-1)} &= \alpha u + \alpha u' + \alpha u'' + \ldots\ldots + \alpha u^{(n)}.
\end{aligned}$$

Since u, u', u'', ... are constant multiples of A, A', A'', ... and the right-hand members of the last n equations are constant multiples of $Y^{(n)}$, Y, Y', ... and therefore of $X^{(n)}$, X, X', ..., the truth of the proposition is established.

Ex. Having given the equations to the sides of any quadrilateral, to find the equations to the sides of an inscribed quadrilateral the opposite sides of which intersect in two given points.

Let the equations to the sides of the original quadrilateral be

$$k = 0, \quad l = 0, \quad m = 0, \quad n = 0.$$

Then, since u, u', u'', u''', must be some constant multiples of k, l, m, n, respectively, the equations to the sides of the inscribed quadrilateral will be of the forms

$$\alpha k + \beta l + \gamma m + \delta n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\alpha(k + \beta l) + \gamma m + \delta n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\alpha(k + \beta l + \gamma m) + \delta n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$k + \beta l + \gamma m + \delta n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let the values of k, l, m, n, be k_1, l_1, m_1, n_1, and k_2, l_2, m_2, n_2, at the two given points.

Then we shall have, for the determination of the constants α, β, γ, γ,

$$\alpha k_1 + \beta l_1 + \gamma m_1 + \delta n_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\alpha(k_2 + \beta l_2) + \gamma m_2 + \delta n_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\alpha(k_1 + \beta l_1 + \gamma m_1) + \delta n_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$k_2 + \beta l_2 + \gamma m_2 + \delta n_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From (5) and (7) we have

$$\beta l_1 + \gamma m_1 = 0, \quad \alpha k_1 + \delta n_1 = 0;$$

and, from (6) and (8),

$$k_2 + \beta l_2 = 0, \quad \gamma m_2 + \delta n_2 = 0.$$

Substituting the values of α, β, γ, δ, given by these four equations, in (1), (2), (3), (4), we shall obtain, for the required equations to the sides of the inscribed quadrilateral,

$$\frac{n_1 k - k_1 n}{n_2 k_1} + \frac{l_1 m - m_1 l}{l_1 m_2} = 0,$$

$$\frac{k_2 l - l_2 k}{k_1 l_2} + \frac{m_2 n - n_2 m}{m_2 n_1} = 0,$$

$$\frac{l_1 m - m_1 l}{l_2 m_1} + \frac{n_1 k - k_1 n}{n_1 k_2} = 0,$$

$$\frac{m_2 n - n_2 m}{m_1 n_2} + \frac{k_2 l - l_2 k}{k_2 l_1} = 0.$$

Cambridge and Dublin Mathematical Journal, for May 1850.

15. If the lines $u = 0$, $v = 0$, be parallel, to shew that, k being an arbitrary constant,

$$u + kv = 0$$

is parallel to each of them.

16. If, through the three angles of a triangle, straight lines be drawn at right angles to the base, to prove that, the general type of the equation to a straight line being

$$(x_2 - x_1)\, y - (y_2 - y_1)\, x = x_2 y_1 - x_1 y_2,$$

the equations to the three sides of the triangle and to the three perpendiculars may be represented by the system of equations

$$u = 0, \quad v = 0, \quad w = 0,$$

$$v - w = 0, \quad w - u = 0, \quad u - v = 0.$$

17. A straight line passes through the intersection of two lines

$$u = 0, \quad v = 0;$$

and through that of two lines

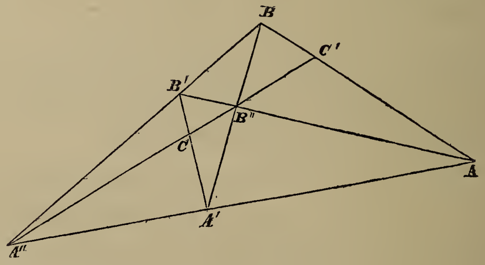
$$u' = 0, \quad v' = 0.$$

To prove that it is impossible to express generally the equation to this fifth line in terms of u, v, u', v'.

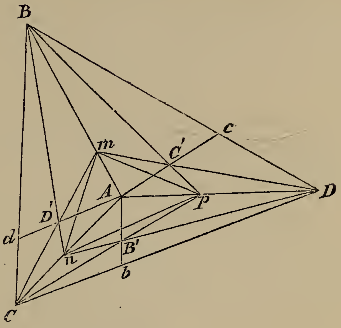
18. If lines be drawn through any two of the points A, B, C, &c., and other lines through any two of the points a, b, c, &c.; to prove that the intersections of AB with ab, of AC with ac, &c. will lie in one straight line, provided that the lines through the intersections of any two of the first series of lines and the corresponding intersections of the second series all pass through the same point.

19. The four angles of a quadrilateral $ABCD$ are bisected by four straight lines: the bisectors of A, B, meet in α; of B, C, in β; of C, D, in γ; and of D, A, in δ. To prove that the directions of $\beta\delta$, $\alpha\gamma$, pass through the intersections of the directions of AD, BC, and of AB, CD, respectively.

20. In any quadrilateral $ABB'A'$, let the diagonals AB', $A'B$, be drawn, intersecting in B'', and the sides AA', BB', produced



to meet in A'': also let $A''B''$ be drawn, cutting AB and $A'B'$ in C' and C respectively. To prove that, α being the intersection of the diagonals of the quadrilateral $AC'CA'$, and β that of the intersection of the diagonals of the quadrilateral $BC'CB'$, the three points α, B'', β, will lie in one straight line.



21. A is any point whatever in the plane of a triangle BCD: from A are drawn to B, C, D, the three straight lines AB, AC, AD. On AB, AC, AD, are taken any points m, n, p, respectively; and Bn, Cm, Cp, Dn, Dm, Bp, are joined: through the points of intersection D', B', C' are drawn the transversals Ad, Ab, Ac.

To prove that

(1). The transversals Dd, Bb, Cc, cut each other in a single point A'.

(2). The four transversals AA', BB', CC', DD', pass all through a single point.

This is a particular case of a theorem in Carnot's *Essai sur la Théorie des Transversales*.

Cambridge and Dublin Mathematical Journal, for Feb. 1850.

SECTION VII.

Rectilinear Areas.

1. To find the area of the figure included by the lines defined by the equations

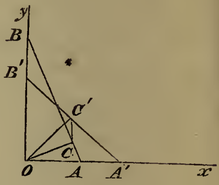
$$x-y=0, \quad x+y=0, \quad x-y=a, \quad x+y=b.$$

The construction of the four lines shews that they form a rectangle, the first and third being two parallel sides. Now the distance between the first and third, which is the same thing as the distance of the origin from the third, is equal to $\dfrac{a}{\sqrt{2}}$. Similarly, the distance between the second and fourth is equal to $\dfrac{b}{\sqrt{2}}$. Hence the area of the figure is equal to $\frac{1}{2}ab$.

2. From the origin of coordinates, O, perpendiculars OC, OC', are let fall upon two straight lines

$$\frac{x}{\alpha}+\frac{y}{\beta}=1, \qquad \frac{x}{\alpha'}+\frac{y}{\beta'}=1,$$

respectively: the points C, C', are joined: to find the area of the triangle COC'.



It is plain that

$$\begin{aligned}\sin\angle COC' &= \sin\angle C'OA'.\cos\angle COA\\ &\quad -\sin\angle COA\cos\angle C'OA',\\ &= \cos A'\sin A-\cos A\sin A',\\ &= \frac{\alpha'}{(\alpha'^2+\beta'^2)^{\frac{1}{2}}}\cdot\frac{\beta}{(\alpha^2+\beta^2)^{\frac{1}{2}}}-\frac{\alpha}{(\alpha^2+\beta^2)^{\frac{1}{2}}}\cdot\frac{\beta'}{(\alpha'^2+\beta'^2)^{\frac{1}{2}}},\\ &= \frac{\alpha'\beta-\alpha\beta'}{(\alpha^2+\beta^2)^{\frac{1}{2}}.(\alpha'^2+\beta'^2)^{\frac{1}{2}}}.\end{aligned}$$

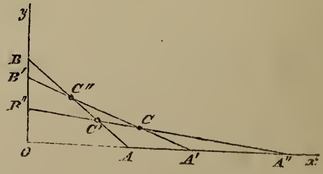
Also $$CO = OA\,.\frac{OB}{AB}=\frac{\alpha\beta}{(\alpha^2+\beta^2)^{\frac{1}{2}}},$$

and, similarly, $$C'O=\frac{\alpha'\beta'}{(\alpha'^2+\beta'^2)^{\frac{1}{2}}}.$$

Hence $$\text{area } COC' = \tfrac{1}{2}\,\frac{\alpha\beta\alpha'\beta'\,(\alpha'\beta-\alpha\beta')}{(\alpha^2+\beta^2)\,(\alpha'^2+\beta'^2)}.$$

3. To find the area $CC'C''$, included between the three lines AB, $A'B'$, $A''B''$, the equations to which are respectively

$$\frac{x}{a}+\frac{y}{b}=1, \qquad \frac{x}{a'}+\frac{y}{b'}=1, \qquad \frac{x}{a''}+\frac{y}{b''}=1.$$



From the geometry we see that

$$CC'C'' = AC''A' - AC'CA',$$
$$= AC''A' + A'CA'' - A''C'A.$$

Now area $AC''A' = \frac{1}{2}AA' \times$ ordinate of C''

$$= \tfrac{1}{2}(a' - a) \,.\, \frac{\dfrac{1}{a} - \dfrac{1}{a'}}{\dfrac{1}{ab'} - \dfrac{1}{a'b}}$$

$$= \tfrac{1}{2}(a' - a)^2 \,.\, \frac{bb'}{a'b - ab'}\,.$$

In the same way it may be proved that

$$\text{area } A'CA'' = \tfrac{1}{2}(a'' - a')^2 \,.\, \frac{b'b''}{a''b' - a'b''}\,,$$

and

$$\text{area } A''C'A = \tfrac{1}{2}(a'' - a)^2 \,.\, \frac{bb''}{a''b - ab''}\,.$$

Hence area $CC'C''$ is equal to

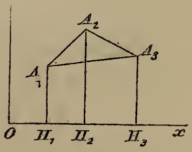
$$\tfrac{1}{2}\left\{\frac{(a' - a)^2}{\dfrac{a'}{b'} - \dfrac{a}{b}} + \frac{(a'' - a')^2}{\dfrac{a''}{b''} - \dfrac{a'}{b'}} + \frac{(a - a'')^2}{\dfrac{a}{b} - \dfrac{a''}{b''}}\right\}.$$

This area may similarly be shewn to be equal also to

$$\tfrac{1}{2}\left\{\frac{(b' - b)^2}{\dfrac{b}{a} - \dfrac{b'}{a'}} + \frac{(b'' - b')^2}{\dfrac{b'}{a'} - \dfrac{b''}{a''}} + \frac{(b - b'')^2}{\dfrac{b''}{a''} - \dfrac{b}{a}}\right\}.$$

4. To find the area of a triangle, the coordinates of the angular points of which are given.

Fig. 1.



Let $A_1A_2A_3$ (fig. 1) be the triangle, the coordinates of A_1, A_2, A_3, being respectively (x_1, y_1), (x_2, y_2), (x_3, y_3).

Then the area of the triangle is equal to

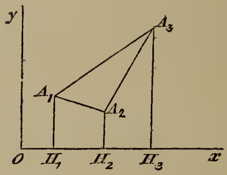
$$A_1A_2H_2H_1 + A_2A_3H_3H_2 - A_1A_3H_3H_1$$

$$= \tfrac{1}{2}(x_2 - x_1)\,(y_2 + y_1) + \tfrac{1}{2}(x_3 - x_2)\,(y_3 + y_2) - \tfrac{1}{2}(x_3 - x_1)\,(y_3 + y_1),$$

whence, P denoting the required area,

$$2P = x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1.$$

Fig. 2.



Had the figure been such as fig. (2), we might have shewn in a similar way that

$$2P = x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3.$$

Thus we see that the area of any triangle, the angular coordinates of which are (x_1, y_1), (x_2, y_2), (x_3, y_3), is equal to

$$\pm \tfrac{1}{2}\{x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1\},$$

the + or − sign taken accordingly as the point A_2 and the indefinite point x are on the opposite or on the same side of the indefinite line through A_1, A_3.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 115, *troisième edition.*

5. To find the area of a polygon the angular coordinates of which are (x_1, y_1), (x_2, y_2), (x_3, y_3), ...(x_n, y_n).

fig. 1.

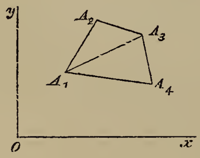


fig. 2.

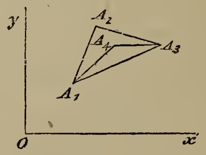
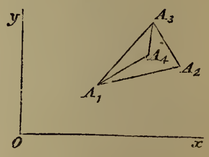


fig. 3.



Taking a quadrilateral, we have, for the area of the triangle $A_1A_2A_3$, in fig. (1) and (2), the expression

$$\tfrac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1),$$

and, in fig. (3), the expression

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

Again, the area of the triangle $A_1A_3A_4$ is equal, in fig. (1) and (3), to the expression

$$\tfrac{1}{2}(x_3y_1 - x_1y_3 + x_4y_3 - x_3y_4 + x_1y_4 - x_4y_1);$$

and, in fig. (2), to the expression

$$\tfrac{1}{2}(x_1y_3 - x_3y_1 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4).$$

Hence the quadrilateral area $A_1A_2A_3A_4$ is equal, in fig. (1) and (2), to the expression

$$\tfrac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_4y_3 - x_3y_4 + x_1y_4 - x_4y_1),$$

and, in fig. (3), to this expression affected by the negative sign.

Thus we see that the area of a quadrilateral $A_1A_2A_3A_4$ is equal to

$$\pm\tfrac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_4y_3 - x_3y_4 + x_1y_4 - x_4y_1),$$

the + or the − sign being taken accordingly as the point A_2 and the indefinite point x are on the opposite or on the same side of the indefinite line through A_1, A_3.

Proceeding in the same way, we shall evidently obtain, for the area of the proposed polygon, the expression

$$\pm\tfrac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_4y_3 - x_3y_4 + \dots\dots + x_1y_n - x_ny_1).$$

De Stainville: *Gergonne, Annales de Mathématiques*, tom. I. p. 190.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 116, *troisième édition.*

6. To find the area of the quadrilateral the equations to the sides of which, taken in order, are

$$y = 0 \dots\dots\dots\dots (1),$$

$$y = x \dots\dots\dots\dots (2),$$

$$(y - a)\sqrt{3} = x - a \dots\dots\dots\dots (3),$$

$$y = x - c \dots\dots\dots\dots (4).$$

At the intersections of (1) and (2), (2) and (3), (3) and (4), (4) and (1), respectively, we shall have

$$\begin{pmatrix} x_1 = 0 \\ y_1 = 0 \end{pmatrix}, \quad \begin{pmatrix} x_2 = a \\ y_2 = a \end{pmatrix}, \quad \left\{\begin{aligned} x_3 &= a + \frac{c\sqrt{3}}{\sqrt{3} - 1} \\ y_3 &= a + \frac{c}{\sqrt{3} - 1} \end{aligned}\right\}, \quad \begin{pmatrix} x_4 = c \\ y_4 = 0 \end{pmatrix}.$$

But, P denoting the required area,

$$2P = x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_4y_3 - x_3y_4 + x_1y_4 - x_4y_1:$$

hence, putting for the coordinates in this expression their proper values, we have

$$2P = a\left(a + \frac{c\sqrt{3}}{\sqrt{3}-1}\right) - a\left(a + \frac{c}{\sqrt{3}-1}\right) + c\left(a + \frac{c}{\sqrt{3}-1}\right).$$

$$= 2ac + \frac{c^2}{\sqrt{3}-1},$$

$$P = ac + \tfrac{1}{4}c^2(\sqrt{3}+1).$$

7. To find the area included between the lines

$$x = 0, \quad y = 0, \quad \frac{x}{2} + \frac{y}{3} = 1.$$

Required area = 3.

8. To find the area of the triangle formed by the three lines

$$x = a, \quad y = mx, \quad y = -mx.$$

Required area = ma^2.

9. To find the area included between the four lines

$$x = a, \quad x = a', \quad y = b, \quad y = b'.$$

Required area = $(a' - a).(b' - b)$.

10. To find the area of the triangle included by the three lines

$$y = b, \quad y = \alpha x, \quad y = \alpha' x.$$

Required area = $\frac{1}{2}b^2\left(\frac{1}{\alpha} \sim \frac{1}{\alpha'}\right)$.

11. To find the area of the triangle formed by the line

$$x + y = 1,$$

a perpendicular upon it from the origin, and the axis of x.

Required area = $\frac{1}{4}$.

12. To find the area included between the three lines

$$x = 0, \quad \alpha x + y = \beta, \quad 2\alpha x + y = 2\beta.$$

Required area = $\frac{\beta^2}{2\alpha}$.

13. To find the area of the triangle included between the three lines $x + 2y = 5, \quad 2x + y = 7, \quad y = x + 1.$

Required area $= \frac{3}{2}$.

14. To find the area included between the lines of which the equations are

$$\frac{x}{\alpha} = \frac{y}{\beta}, \qquad \frac{x}{\alpha'} = \frac{y}{\beta'}, \qquad \frac{x}{a} + \frac{y}{b} = 1.$$

The required area is equal to

$$\tfrac{1}{2}a^2b^2 . \frac{\alpha\beta' \sim \alpha'\beta}{(a\beta' + b\alpha').(a\beta + b\alpha)}.$$

15. To find the area included between the lines of which the equations are

$$x = 0, \quad y = 0, \quad x = a, \quad y = b.$$

The area is equal to ab.

16. To find the area of the quadrilateral included between the four lines $x = a, \quad x = a', \quad y = \alpha x, \quad y = \alpha' x.$

Required area $= \frac{1}{2}(a'^2 \sim a^2).(\alpha' \sim \alpha)$.

17. To find the area included between the lines of which the equations are

$$x = 0, \quad y = 0, \quad \frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1.$$

The area is equal to

$$\tfrac{1}{2}(ab \sim a'b').$$

18. To find the area of a parallelogram, the equations to the sides of which are

$$lx + my \pm \delta = 0,$$
$$l'x + m'y \pm \delta' = 0,$$

ω being the angle between the coordinate axes.

The required area is equal to

$$\frac{4\delta\delta' \sin\omega}{lm' \sim l'm}.$$

19. To find the area of the quadrilateral included between the four lines

$$y = \alpha x, \quad y = \alpha' x, \quad y = mx + c, \quad y = m'x + c'.$$

Required area $= \pm \frac{1}{2} (\alpha' - \alpha) . \left\{ \frac{c'^2}{(\alpha - m').(\alpha' - m')} - \frac{c^2}{(\alpha - m).(\alpha' - m)} \right\}$.

20. To find the area of the quadrilateral included between the four lines

$$x + y = 4, \quad -3x + 2y = 5, \quad 12x + 7y = -35, \quad 4x - 5y = 25.$$

Required area = 30·8.

21. If x_1, y_1; x_2, y_2; x_3, y_3; ......(x_n, y_n); denote the coordinates of the angular points of a polygon taken in order, the axes being inclined to each other at any angle ω, to prove that the area of the polygon is equal to

$$\pm \tfrac{1}{2} \sin \omega \, \{x_2 y_1 - x_1 y_2 + x_3 y_2 - x_2 y_3 + x_4 y_3 - x_3 y_4 + \ldots + x_1 y_n - x_n y_1\}.$$

22. To find the equations to straight lines passing through a given point and including a given triangular area between given axes of coordinates.

Let a, b, be the coordinates of the given point, α the angle between the axes, and c^2 the given area. Then the equations to the lines, which satisfy the conditions of the problem, will be

$$\pm 2c^2 (x - a)(y - b) + \sin \alpha \, \{a (y - b) - b (x - a)\}^2 = 0.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 139, *troisième édition.*

23. To bisect a trapezium $AOBC$, the angles A, O, of which are right angles, by a straight line at right angles to OA.

Let $OA = a$, $CA = h$, $BO = h'$, and, the lines OA, OB, being taken as axes of x, y, respectively, let x be the abscissa of the bisecting line. Then

$$x = \frac{a}{h - h'} \left\{ - h' \pm \left(\frac{h^2 + h'^2}{2} \right)^{\frac{1}{2}} \right\}.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 160, *troisième édition.*

24. A, B, are two given points; P is a point so placed that the sum of the squares of the distances PA, PB, bears to the area of the triangle APB the ratio of f to g, and that the angle APB has a given value. Supposing the middle point O between A and B to be the origin of rectangular coordinates, OA produced indefinitely being the axis of x, to find the coordinates of P.

Let $OA = a = OB$, $\angle APB = \alpha$. Then

$$x = \pm \frac{a}{f \sin \alpha - 4g \cos \alpha} \cdot (f^2 \sin^2 \alpha - 16g^2)^{\frac{1}{2}},$$

$$y = \frac{4ag \sin \alpha}{f \sin \alpha - 4g \cos \alpha}.$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 362.

25. To find the lengths of the sides AA', $A'A''$, $A''A$, of a triangle and its area, the equations to its sides AA', $A'A''$, $A''A$, being respectively

$$x \cos \alpha'' + y \sin \alpha'' = \delta'', \quad x \cos \alpha + y \sin \alpha = \delta, \quad x \cos \alpha' + y \sin \alpha' = \delta'.$$

The required lengths are

$$A'A = \frac{\delta'' \sin (\alpha - \alpha') + \delta \sin (\alpha' - \alpha'') + \delta' \sin (\alpha'' - \alpha)}{\sin (\alpha - \alpha'').\sin (\alpha' - \alpha'')},$$

$$A'A'' = \frac{\delta \sin (\alpha' - \alpha'') + \delta' \sin (\alpha'' - \alpha) + \delta'' \sin (\alpha - \alpha')}{\sin (\alpha' - \alpha).\sin (\alpha'' - \alpha)}$$

$$A''A = \frac{\delta' \sin (\alpha'' - \alpha) + \delta'' \sin (\alpha - \alpha') + \delta \sin (\alpha' - \alpha'')}{\sin (\alpha'' - \alpha').\sin (\alpha - \alpha')},$$

and twice the area of the triangle is equal to

$$\frac{\{\delta \sin (\alpha' - \alpha'') + \delta' \sin (\alpha'' - \alpha) + \delta'' \sin (\alpha - \alpha')\}^2}{\sin (\alpha - \alpha').\sin (\alpha' - \alpha'').\sin (\alpha'' - \alpha)}.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 116.

CIRCLE.

SECTION I.

Referred to two Perpendicular Diameters. Tangents.

1. To find the relation among the quantities a, b, c, that the line

$$\frac{x}{a} + \frac{y}{b} = 1$$

may be a tangent to the circle

$$x^2 + y^2 = c^2.$$

If α be the inclination of the radius of any point in the circumference to the axis of x, the equation to the corresponding tangent is

$$x \cos \alpha + y \sin \alpha = c.$$

In order that this equation may coincide with that to the given straight line, we must have

$$\frac{\cos \alpha}{c} = \frac{1}{a}, \qquad \frac{\sin \alpha}{c} = \frac{1}{b}.$$

Squaring and then adding these equations, we get, for the required relation,

$$\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

2. Tangents are drawn to a circle

$$x^2 + y^2 = c^2$$

at two points (x', y'), (x'', y''): to find the distance of a point (h, k) from a line passing through the centre of the circle and the intersection of the two tangents.

The equations to the two tangents are

$$xx' + yy' = c^2,$$
$$xx'' + yy'' = c^2.$$

Subtracting one of these equations from the other, we get for the equation to the line, which passes through the centre and the intersection of the two tangents,

$$x(x' - x'') + y(y' - y'') = 0.$$

Let δ denote the distance of (h, k) from this line; then

$$\delta^2 = \frac{\{h(x' - x'') + k(y' - y'')\}^2}{(x' - x'')^2 + (y' - y'')^2}.$$

3. To find the equations to straight lines touching the circle

$$x^2 + y^2 = 10$$

at points the common abscissa of which is unity.

The required equations are

$$x \pm 3y = 10.$$

4. To find the equation to a circle, the centre of which is at the origin of rectangular coordinates, and which is touched by the line

$$y = 2x + 3.$$

The required equation is

$$x^2 + y^2 = \tfrac{9}{5}.$$

5. To find the intercepts on the axes of coordinates of the tangent to a circle drawn parallel to a given straight line.

If c be the radius of the circle, and l, m, the direction-cosines of the given line, the required intercepts are equal to $\frac{c}{l}$, $\frac{c}{m}$.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 148, *troisième édition.*

6. If, from a given point S, a perpendicular SY be drawn to the tangent PY at any point P of a circle of which the centre is C, and, in the line MP at right angles to CS and produced if necessary, a point Q be always taken such that $MQ = SY$ to find the locus of Q.

Let C be the origin of coordinates, CS, produced indefinitely, being the axis of x; then, denoting CS by a, and the radius by r, the equation to the locus of Q will be

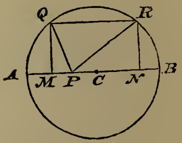
$$ax - ry = r^2.$$

SECTION II.

Referred to two Perpendicular Diameters. Chords.

1. If a chord be drawn parallel to a diameter of a circle, and from any point in the diameter lines be drawn to its extremities; the sum of their squares will be equal to the sum of the squares of the segments of the diameter.

Let the chord QR be drawn parallel to the diameter AB; let Q, R, be joined by straight lines to any point P in AB. Draw QM, RN, at right angles to AB.



Let $QM = y = RN$, $CP = a$, $r =$ the radius of the circle, $CM = x = CN$. Then

$$\begin{aligned} PQ^2 + PR^2 &= y^2 + (x - a)^2 + y^2 + (x + a)^2 \\ &= 2(x^2 + y^2 + a^2) \\ &= 2(r^2 + a^2) \\ &= (r - a)^2 + (r + a)^2 \\ &= PA^2 + PB^2. \end{aligned}$$

2. To prove analytically that the angle in a semicircle is a right angle.

Let (a, b), be the coordinates of one end of the diameter of the semicircle; $(-a, -b)$ will be those of the other end.

The equations to the two chords will be

$$\frac{x - a}{l} = \frac{y - b}{m},$$

$$\frac{x + a}{l'} = \frac{y + b}{m'}.$$

At the intersection of these two lines

$$\frac{x^2 - a^2}{ll'} = \frac{y^2 - b^2}{mm'} \quad \text{..................... (1).}$$

But, c being the radius of the circle,

$$x^2 + y^2 = c^2,$$

$$a^2 + b^2 = c^2,$$

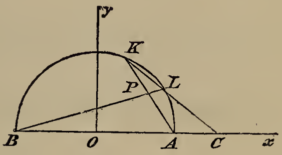
and therefore $$x^2 - a^2 + y^2 - b^2 = 0 \quad \text{.................. (2).}$$

From (1) and (2) we see that

$$ll' + mm' = 0,$$

the condition for the perpendicularity of the two chords.

3. C is a fixed point in the diameter BA, produced, of a circle; from C any line CLK is drawn cutting the circle in two points L, K; AK, BL, are joined, intersecting in P; to find the locus of P.



Let OCx be taken as the axis of x, and Oy, at right angles to it, as that of y. Let a denote the radius of the circle.

Then the equations to AK, BL, will be, respectively, of the forms

$$y = m(x - a),$$

$$y = n(x + a);$$

both of these equations being comprehended in the following one,

$$(y - mx + ma) . (y - nx - na) = 0,$$

or $$y^2 + mnx^2 - (m + n)xy + a(m - n)y - mna^2 = 0 \text{...(1).}$$

The equation to the circle is

$$x^2 + y^2 = a^2 \quad \text{........................(2).}$$

Eliminating y^2 between (1) and (2), we get, relatively to the points of intersection of AK, BL, with the circle,

$$y^2 - mny^2 - (m + n)xy + a(m - n)y = 0;$$

whence, dividing by y, we have

$$(1 - mn)y - (m + n)x + (m - n)a = 0 \quad \text{...... (3),}$$

as the equation to the chord KL.

Let $OC = c$; then, since KL, produced, passes through C, we have, from (3),

$$(m - n)\, a = (m + n)\, c \quad\text{..................(4).}$$

At the point P, in which AK, BL, intersect, we have

$$m\,(x - a) = n\,(x + a) \quad\text{.................. (5).}$$

From (4) and (5) there is

$$a\,\{(x + a) - (x - a)\} = c\,\{(x + a) + (x - a)\},$$

whence $$x = \frac{a^2}{c},$$

which is the equation to the locus of P, being therefore a straight line at right angles to the axis of x, passing through the point at which a tangent from C touches the circle.

4. To find the length of a chord of the circle

$$x^2 + y^2 = c^2,$$

the equation to the chord being

$$\frac{x}{a} + \frac{y}{b} = 1.$$

The required length $= 2\left(c^2 - \dfrac{a^2b^2}{a^2 + b^2}\right)^{\frac{1}{2}}$.

5. If 2α, $2\alpha'$, be the inclinations of two radii of a circle

$$x^2 + y^2 = c^2$$

to the axis of x, to find the equation to the chord joining the extremities of the radii.

The required equation is

$$x \cos(\alpha + \alpha') + y \sin(\alpha + \alpha') = c \cos(\alpha - \alpha').$$

6. From any point Q in the circumference of a semicircle two chords are drawn to the extremities A, B, of its diameter. From any point M in this diameter a perpendicular to it is drawn, which cuts AQ in H, BQ in K, and the curve in P. To prove that $$MH \,.\, MK = (MP)^2.$$

7. If H represent the harmonic mean between the abscissæ, and K that between the ordinates, of the points, in which a circle

$$x^2 + y^2 = c^2$$

is cut by a chord $$lx + my = \delta,$$

where l, m, are the direction-cosines; to prove that

$$Hl + Km = \delta - \frac{c^2}{2\delta}.$$

8. To determine, in the circumference of a given circle, a point P, such that, if it be joined to two given points A, B, the lines AP, BP, may intersect the circumference in the extremities A', B', of a chord $A'B'$ parallel to AB.

The equation to the circle being

$$x^2 + y^2 = c^2,$$

and (α, β), (α', β'), being the coordinates of A, B, respectively, the required point P will be determined by the intersection of the given circle with the straight line defined by the equation

$$\frac{\alpha x + \beta y - c^2}{\alpha^2 + \beta^2 - c^2} = \frac{\alpha' x + \beta' y - c^2}{\alpha'^2 + \beta'^2 - c^2}.$$

This is a particular instance of the more general problem, "To inscribe in a given conic section a triangle the sides of which are subjected to pass through three given points."

Lamé: *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*, p. 74.

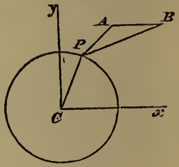
SECTION III.

Referred to two Perpendicular Diameters. Points.

1. Having given two points A and B, to find the position of a point P in the circumference of a given circle, such that, if AP, BP, be joined, and P be joined to the centre C of the circle,

$$\frac{\sin CPA}{\sin CPB} = \frac{m}{n}.$$

Let Cx, parallel to AB, be taken as the axis of x, and Cy, at right angles to Cx, as that of y.



Let (α, β), (α', β), be the coordinates of A, B, respectively, and x, y, those of the required point P. Let c be the radius of the circle.

Then the equation for determining x or y will be an equation of the sixth degree resulting from the elimination of y or x between the two equations

$$x^2 + y^2 = c^2,$$

and $$n \,.\, (\alpha' y - \beta x) \left\{(\alpha - x)^2 + (\beta - y)^2\right\}^{\frac{1}{2}}$$
$$= m \,.\, (-\alpha y + \beta x) \left\{(\alpha' - x)^2 + (\beta - y)^2\right\}^{\frac{1}{2}}.$$

Descartes: *Lettres*, Lettre 65, tom. III. p. 357.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 168, *troisième édition.*

2. The distances of a given point C from two points P, Q, in the circumference of a circle

$$x^2 + y^2 = r^2,$$

are each equal to nr: to find the equations to the two lines PC, QC.

If (a, b) be the coordinates of C, the equations to PC, QC, are included in the equation

$$\left\{a^2 + b^2 + (n^2 - 1)\, r^2\right\}^2 . \left\{(x - a)^2 + (y - b)^2\right\}$$
$$= 4n^2r^2 . \left\{a\,(x - a) + b\,(y - b)\right\}^2.$$

SECTION IV.

Referred to any Rectangular Axes.

1. To determine the relation among the constants of the two equations

$$lx + my = \delta,$$
$$(x - a)^2 + (y - b)^2 = c^2,$$

in order that they may touch each other

The coordinates of the centre of the circle are a, b, and therefore the square of the distance of the centre of the circle from the straight line is equal to

$$\frac{(al + bm - \delta)^2}{l^2 + m^2}.$$

But this distance, the line being a tangent, must be equal to the radius: hence

$$(al + bm - \delta)^2 = (l^2 + m^2)\, c^2,$$

which is the required relation.

Cor. If l, m, be the direction-cosines of the straight line,

$$l^2 + m^2 = 1,$$

and the relation is reduced to

$$\pm\, (al + bm - \delta) = c.$$

2. To find the equation to a circle having for a diameter the line joining the points of intersection of the line $y = \alpha x$ and the circle

$$y^2 = 2cx - x^2.$$

At the intersections of the line and circle

$$\alpha^2 x^2 = 2cx - x^2,$$

and therefore the coordinates of these two points are

$$\begin{pmatrix} x_1 = 0 \\ y_1 = 0 \end{pmatrix}$$

and

$$\left[\begin{matrix} x_2 = \dfrac{2c}{1 + \alpha^2} \\[2ex] y_2 = \dfrac{2\alpha c}{1 + \alpha^2} \end{matrix}\right].$$

Let x', y', be the coordinates of the centre of the new circle, and c' its radius. Then

$$x' = \tfrac{1}{2} x_2 = \frac{c}{1 + \alpha^2},$$

$$y' = \tfrac{1}{2} y_2 = \frac{\alpha c}{1 + \alpha^2},$$

and

$$c'^2 = x'^2 + y'^2 = \frac{c^2}{1 + \alpha^2}.$$

Hence the equation to the new circle is

$$\left(x - \frac{c}{1+\alpha^2}\right)^2 + \left(y - \frac{\alpha c}{1+\alpha^2}\right)^2 = \frac{c^2}{1+\alpha^2},$$

or $$(1+\alpha^2).(x^2+y^2) = 2c\,(x+\alpha y).$$

3. A straight line cuts two given circles in such a manner that the parts of it intercepted by them are of given lengths: to find the point in which this line cuts the line joining the centres of the two circles.

Let the line joining the centres of the circles be taken as the axis of x, and one of their centres as the origin of coordinates. Their equations will then be of the forms

$$x^2 + y^2 = c^2 \text{ .... } (1),$$

$$(x+a)^2 + y^2 = c'^2 \text{ .... } (2).$$

The chords intercepted by the two given circles being equal and known, the perpendiculars from the centres of the two circles upon the intersecting line must also be given: let them be represented respectively by p, p'. Let the equation to the intersecting line be

$$\frac{x}{m} + \frac{y}{n} = 1 \text{ .... } (3).$$

Then, p denoting the distance of the centre of (1) from (3),

$$p = \frac{1}{\left(\frac{1}{m^2} + \frac{1}{n^2}\right)^{\frac{1}{2}}};$$

and, p' denoting the distance of $(-a, 0)$, the centre of (2), from (3),

$$\pm p' = \frac{\frac{a}{m} + 1}{\left(\frac{1}{m^2} + \frac{1}{n^2}\right)^{\frac{1}{2}}}.$$

Hence $$\pm\frac{p'}{p} = \frac{a}{m} + 1,$$

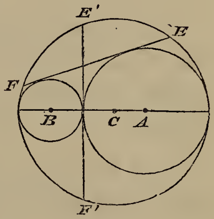
$$\therefore\ m = \frac{ap}{\pm p' - p},$$

which determines the distance of the required point from the origin.

Lacroix: *Application de l'Algèbre à la Géométrie*, No. 109.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 152, *troisième édition.*

4. Diameters of two circles, the centres of which are A, B, make up a diameter of a greater circle, the centre of which is C.



To compare the lengths of the two chords EF, $E'F'$, of the greater circle, which touch both the smaller ones.

Let a, b, be the radii of the two smaller circles, $a + b$ being therefore the radius of the larger. Let $EF = c$, $E'F' = c'$.

Let the equation to EF be

$$\frac{x}{m} + \frac{y}{n} = 1.$$

Then, p denoting the distance of C from EF,

$$\tfrac{1}{4}c^2 = (a+b)^2 - p^2$$

$$= (a+b)^2 - \frac{m^2n^2}{m^2+n^2} \quad \dots\dots\dots\dots\dots\dots(1).$$

Again, the distances of the centres of the circles A, B, from EF, being a, b, respectively, we have

$$a = \frac{n(b-m)}{(m^2+n^2)^{\frac{1}{2}}}, \qquad b = -\frac{n(a+m)}{(m^2+n^2)^{\frac{1}{2}}}.$$

From these two equations we may, by a few obvious algebraical processes, ascertain that

$$m^2 = \left(\frac{a^2+b^2}{a-b}\right)^2, \qquad n^2 = \frac{(a^2+b^2)^2}{4ab}.$$

Hence, substituting these values of m^2 and n^2 in (1), and reducing, we shall get

$$c^2 = 16ab \,.\, \frac{a^2+ab+b^2}{(a+b)^2}.$$

Again, c' denoting the chord $E'F'$, we have

$$\tfrac{1}{4}c'^2 = 2a.2b,$$

or

$$c'^2 = 16ab.$$

Hence the ratio between the lengths of the chords EF, $E'F'$, is equal to

$$\frac{(a^2 + ab + b^2)^{\frac{1}{2}}}{a + b}.$$

5. If three fixed circles be cut by any other, to prove that the three chords will form a triangle, the loci of the angular points of which are three fixed straight lines meeting in a point.

Let the coordinates of the centres of the three fixed circles be (a, b), (a', b'), (a'', b''), and their respective radii c, c', c''. Let (p, q) be the centre of the variable circle, and r its radius.

The equations to the first of the fixed circles will be

$$(x - a)^2 + (y - b)^2 = c^2,$$

and to the variable circle

$$(x - p)^2 + (y - q)^2 = r^2.$$

Subtracting the latter equation from the former, we have

$$2x(p - a) + 2y(q - b) + r^2 - c^2 = p^2 - a^2 + q^2 - b^2,$$

which is the equation to the chord of intersection.

The equation to the chord of the second of the fixed circles will be, similarly,

$$2x(p - a') + 2y(q - b') + r^2 - c'^2 = p^2 - a'^2 + q^2 - b'^2.$$

At the intersection of these two chords, there is, one equation being subtracted from the other,

$$2x(a - a') + 2y(b - b') + c^2 - c'^2 = a^2 - a'^2 + b^2 - b'^2,$$

which is the equation to the rectilinear locus of one of the angular points of the triangle of chords.

The equations to the other two loci are, by similar reasoning,

$$2x(a' - a'') + 2y(b' - b'') + c'^2 - c''^2 = a'^2 - a''^2 + b'^2 - b''^2,$$

$$2x(a'' - a) + 2y(b'' - b) + c''^2 - c^2 = a''^2 - a^2 + b''^2 - b^2.$$

If we add together the equations to these three loci, we shall evidently obtain an identical equation. They must therefore pass through a single point.

6. To find the equation to a circle the diameter of which is the common chord of the two circles

$$x^2 + y^2 = c^2, \qquad (x - a)^2 + y^2 = c^2.$$

The required equation is

$$x^2 - ax + y^2 = c^2 - \tfrac{1}{2}a^2.$$

7. To find the equation to the straight line which passes through the centres of the two circles

$$x^2 + 2x + y^2 = 0, \qquad y^2 + 2y + x^2 = 0.$$

The required equation is

$$x + y + 1 = 0.$$

8. To find the equations to the circles which touch the three lines

$$x = a, \quad y = 2b, \quad y = 2b'.$$

The required equations are

$$(x - a \pm b' \mp b)^2 + (y - b - b')^2 = (b' - b)^2.$$

9. To find the equation to a circle which passes through three points the coordinates of which are (1, 2), (1, 3), (2, 5).

The required equation is

$$x^2 - 9x + y^2 - 5y + 14 = 0.$$

10. To find the equation to a circle, which passes through the three points (0, 0), (– 8a, 0), (0, 6a).

The equation to the circle will be

$$(x + 4a)^2 + (y - 3a)^2 = 25a^2.$$

11. To find the centre and radius of the circle

$$x^2 + y^2 - 6x + 4y + 4 = 0.$$

The coordinates of its centre are (3, – 2), and its radius is 3.

12. To find the equation to the common chord of the two circles

$$(x-1)^2 + (y-2)^2 = 6,$$
$$(x-2)^2 + (y-3)^2 = 8.$$

The required equation is

$$x + y = 3.$$

13. To find the equation to a circle having for diameter the distance of a given point from the origin of coordinates.

If α, β, be the coordinates of the given point, the required equation will be

$$x^2 + y^2 = \alpha x + \beta y.$$

14. To find the length of the common chord of the two circles

$$(x-\alpha)^2 + (y-\beta)^2 = \gamma^2, \quad (x-\beta)^2 + (y-\alpha)^2 = \gamma^2.$$

$$\text{The required length} = \{4\gamma^2 - 2(\alpha-\beta)^2\}^{\frac{1}{2}}.$$

15. To prove that all circles represented by the equation

$$(x-\alpha)^2 + (y-\beta)^2 - \gamma^2 + k\{(x-\alpha_,)^2 + (y-\beta_,)^2 - \gamma_,^2\} = 0,$$

where α, β, γ, $\alpha_,$, $\beta_,$, $\gamma_,$ are constant, and k any quantity whatever, pass through two fixed points; and to find the equation to the straight line which passes through the two points.

The required equation to the straight line will be

$$2(\alpha-\alpha_,)x + 2(\beta-\beta_,)y + \alpha_,^2 + \beta_,^2 - \alpha^2 - \beta^2 = \gamma_,^2 - \gamma^2.$$

16. To find the equation to a straight line passing through the origin of coordinates and touching the circle of which the equation is

$$x^2 + y^2 - 3x + 4y = 0.$$

The required equation is

$$3x = 4y.$$

17. To find the equations to the lines which touch both the circles

$$x^2 + y^2 = c^2, \quad (x-a)^2 + y^2 = c'^2.$$

The required equations to the four common tangents are

$$(c \pm c')x \pm \{a^2 - (c \pm c')^2\}^{\frac{1}{2}} y = ac.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 150, *troisième édition.*

18. The abscissæ of the centres of two circles are α_1, α_2, and the lengths of tangents drawn to them from the origin are l_1, l_2; to find the position of a point in the axis of x such that tangents drawn from it to the two circles shall be equal to each other.

The distance of the required point from the origin is equal to

$$\frac{l_2^2 - l_1^2}{2(\alpha_2 - \alpha_1)}.$$

19. To find the equation to a circle which touches the three lines

$$x = 0, \quad y = 0, \quad \frac{x}{a} + \frac{y}{b} = 1,$$

in the positive quadrant.

The required equation, if we put $a^2 + b^2 = c^2$, is

$$x^2 + y^2 - (a + b - c)(x + y) + (a + b - c)^2 = 0.$$

20. To find the equation to a circle which touches the two lines

$$\frac{x}{a} + \frac{y}{b} = \lambda, \quad \frac{x}{a} + \frac{y}{b} = \mu,$$

and of which the centre lies in the line

$$ax = by.$$

The required equation is

$$\left(\frac{1}{a^2} + \frac{1}{b^2}\right)(x^2 + y^2) - (\lambda + \mu)\left(\frac{x}{a} + \frac{y}{b}\right) + \lambda\mu = 0.$$

21. A fixed circle is cut by a series of circles, all of which pass through two given points; to shew that the straight lines which join the points of intersection of the fixed circle with each of the series, all converge to one point, and to determine the position of this point.

If the equation to the fixed circle be

$$x^2 + y^2 = r^2,$$

and (a, b), (α, β), be the two given points, the coordinates of the required point will be

$$\frac{(\alpha - a)\,r^2 - \alpha\,(a^2 + b^2) + a\,(\alpha^2 + \beta^2)}{\alpha^2 + \beta^2 - a^2 - b^2},$$

and
$$\frac{(\beta - b)\,r^2 - \beta\,(a^2 + b^2) + b\,(\alpha^2 + \beta^2)}{\alpha^2 + \beta^2 - a^2 - b^2}.$$

Leybourn: *Mathematical Repository, New Series*, vol. I. part II. p. 39.

22. If P be the point of intersection (not the origin) of the circles

$$x^2 + y^2 = \frac{2x}{a}, \qquad x^2 + y^2 = \frac{2y}{b},$$

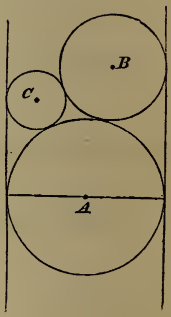
and P' the point of intersection (not the origin) of the circles

$$x^2 + y^2 = \frac{2x}{a'}, \qquad x^2 + y^2 = \frac{2y}{b'};$$

to prove that

$$PP' = \frac{2}{(a^2 + b^2)^{\frac{1}{2}}.(a'^2 + b'^2)^{\frac{1}{2}}} \cdot \{(a - a')^2 + (b - b')^2\}^{\frac{1}{2}}.$$

23. Two parallel tangents are drawn to a given circle A, and two circles B, C, touch each other and A externally, touching also the two tangents respectively; to find the locus of the point of contact of B and C.



Let a denote the radius of A. Then, the centre of A being taken as the origin of coordinates, and the diameter of A, which is perpendicular to the two parallel tangents, as the axis of x, the equation to the required locus will be

$$(x^2 + y^2)^2 = a^2\,(x^2 + 2y^2).$$

SECTION V.

Referred to two Tangents as Axes of Coordinates.

1. To find the equation to a circle referred to two tangents as axes of coordinates.

Let ω denote the inclination of the axes, c the radius of the circle, and a, a, the equal coordinates of its centre.

Then the equation to the circle will be

$$(x-a)^2+(y-a)^2+2(x-a).(y-a).\cos\omega=c^2:$$

but from the geometry it is plain that c is equal to $a\sin\omega$; hence the equation becomes

$$x^2+2xy\cos\omega+y^2-2a(1+\cos\omega).(x+y)+2a^2(1+\cos\omega)=a^2\sin^2\omega,$$

or $$(x+y)^2-2a(1+\cos\omega)(x+y)+a^2(1+\cos\omega)^2$$
$$=2xy(1-\cos\omega),$$

or $$\left(x+y-2a\cos^2\frac{\omega}{2}\right)^2=4\sin^2\frac{\omega}{2}.xy,$$

or $$x+y\pm 2\sin\frac{\omega}{2}.(xy)^{\frac{1}{2}}=2a\cos^2\frac{\omega}{2}.$$

2. To find the equation to a circle, of which the radius is c, and which is referred to two rectangular tangents as axes.

The required equation is

$$x^2+y^2-2c(x+y)+c^2=0.$$

3. To find the equations to two circles which touch the rectangular axes of x and y, and pass through a given point (a, b); to find also the equation to their common chord.

The equations to the two circles are

$$x^2+y^2-2\{a+b\pm(2ab)^{\frac{1}{2}}\}.(x+y)+\{a+b\pm(2ab)^{\frac{1}{2}}\}^2=0,$$

and the equation to the common chord is

$$x+y=a+b.$$

4. Supposing the inclination between the coordinate axes to be $\frac{\pi}{3}$, to prove that

$$x+y-a=(xy)^{\frac{1}{2}}$$

is the equation to a circle the radius of which is equal to $\frac{a}{\sqrt{3}}$.

5. To find the equation to a circle touching a given circle and the two sides of a given angle.

The two sides of the given angle ω being taken as axes of coordinates, let c be the radius and h, k, the coordinates of the centre of the given circle. Then the equation to the required circle will be

$$\left(x + y - 4a\cos^2\frac{\omega}{2}\right)^2 = 4\sin^2\frac{\omega}{2}\,.\,xy,$$

a being determined by the equation

$$(h - a)^2 + (k - a)^2 + 2(h - a)(k - a)\cos\omega = (c \pm a\sin\omega)^2.$$

There being four values of a, it is plain that there will be four circles satisfying the conditions of the problem.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 159, *troisième édition.*

SECTION VI.

Referred to any Oblique Axes.

1. To determine the radius of the circle represented by the equation

$$x^2 + y^2 + 2xy\cos\omega = ax + by.$$

The general equation to the circle referred to oblique axes including an angle ω is

$$(x - h)^2 + (y - k)^2 + 2(x - h)(y - k)\cos\omega = c^2,$$

where c is the radius.

Equating the coefficients of x, y, and the constant terms in these two equations, we have

$$2(h + k\cos\omega) = a \ldots\ldots\ldots\ldots(1),$$

$$2(k + h\cos\omega) = b \ldots\ldots\ldots\ldots(2),$$

$$c^2 = h^2 + k^2 + 2hk\cos\omega \ldots\ldots\ldots\ldots(3).$$

Multiplying (1) by h, (2) by k, and adding the resulting equations, we see that

$$2(h^2 + k^2 + 2hk\cos\omega) = ah + bk,$$

and therefore, by (3),

$$2c^2 = ah + bk \dots\dots\dots\dots\dots\dots\dots\dots (4).$$

From (1) and (2) we readily find that

$$h = \frac{a - b\cos\omega}{2\sin^2\omega}, \qquad k = \frac{b - a\cos\omega}{2\sin^2\omega};$$

and therefore, by (4),

$$4c^2 = \frac{a^2 + b^2 - 2ab\cos\omega}{\sin^2\omega},$$

which determines the magnitude of the radius.

2. To determine the inclination of the coordinate axes in order that the equation

$$x^2 - xy + y^2 - ax - ay = 0$$

may represent a circle; and to find the magnitude of its radius.

The inclination of the axes $= \frac{2}{3}\pi$, and the radius $= a$.

3. The axes Ox, Oy, cut a circle in points A, A', B, B', respectively; to compare the values of x, y, at the intersection of the chords AB', $A'B$.

Let $OA = \alpha$, $OA' = \alpha'$, $OB = \beta$, $OB' = \beta'$; then

$$\frac{x}{y} = \frac{\beta' - \beta}{\alpha' - \alpha}.$$

SECTION VII.

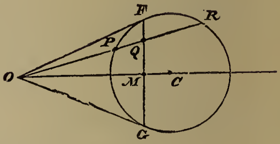
Polar Coordinates.

1. From a point O, without a circle, two tangents OF, OG, are drawn to the curve; the chord FG is joined. From O any straight line $OPQR$ is drawn, cutting the circle in the points P, R, and the chord in Q. To prove that the line OR is harmonically divided in the points P and Q.

Join OC, C being the centre of the circle, cutting FG in M.

Let a = the radius of the circle, $OC = c$.

Let $\angle ROM = \theta$, and let OP or OR be denoted by r.



Then, by the polar equation to the circle, O being the pole and OC the direction of the prime radius vector, there is

$$(r\cos\theta - c)^2 + r^2\sin^2\theta = a^2,$$

or

$$r^2 - 2c\cos\theta . r + c^2 - a^2 = 0,$$

or, since $c^2 - a^2 = f^2$, f denoting OF,

$$r^2 - 2c\cos\theta . r + f^2 = 0.$$

Hence it appears, by the theory of equations, that

$$OP.OR = f^2,$$

and

$$OP + OR = 2c\cos\theta,$$

whence

$$\frac{1}{OP} + \frac{1}{OR} = \frac{2c\cos\theta}{f^2}.$$

But, joining CF, we see by the similar triangles OMF, OFC, that

$$OQ\cos\theta = OM = \frac{f^2}{c}.$$

Hence

$$\frac{1}{OP} + \frac{1}{OR} = \frac{2}{OQ}.$$

De La Hire: *Sectiones Conicæ*, lib. I. prop. 21.

2. If on any three chords, drawn through the same point in the circumference of a circle, as diameters, three circles be described; the points of intersection of these circles lie in a single straight line.

Let the point of the original circle through which the three chords are drawn be taken as the pole and the diameter through it as the prime radius vector.

Let c represent the diameter of the original circle, and c' of one of the three new ones. Let α represent the angle between c and c'.

Then the polar equation to the circle, of which c' is the diameter, will be

$$r = c' \cos(\theta - \alpha):$$

but it is plain that $c' = c \cos\alpha;$

hence $r = c \cos\alpha \cos(\theta - \alpha)$

$$= \tfrac{1}{2}c\{\cos\theta + \cos(\theta - 2\alpha)\}\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly, the equations to the other two new circles will be

$$r = c\cos\alpha' \cos(\theta - \alpha') = \tfrac{1}{2}c\{\cos\theta + \cos(\theta - 2\alpha')\}\ldots(2),$$

$$r = c\cos\alpha'' \cos(\theta - \alpha'') = \tfrac{1}{2}c\{\cos\theta + \cos(\theta - 2\alpha'')\}\ldots(3).$$

At the intersection of (1) and (2),

$$\theta = \alpha + \alpha', \quad r = c\cos\alpha\cos\alpha';$$

this intersection therefore lies in a straight line represented by the equation

$$r\cos(\alpha + \alpha' + \alpha'' - \theta) = c\cos\alpha\cos\alpha'\cos\alpha''.$$

By symmetry it is plain that this line will contain also the intersection of (2), (3), and that of (3), (1).

Greathead: *Cambridge Mathematical Journal*, vol. I. p. 168.

3. To determine the magnitude and position of the circle of which the equation is

$$r^2 - 2(\cos\theta + \sqrt{3}\sin\theta)r = 5.$$

The coordinates of the centre are $\theta = \tfrac{1}{3}\pi$, $r = 2$, and the radius is equal to 3.

4. To prove analytically that if, from a point O without a circle, two straight lines OPQ, OA, be drawn, one of which cuts the circle in P, Q, and the other touches it in A, the square of the line OA which touches the circle is equal to the rectangle contained by the straight line OPQ which cuts the circle and the part of it OP without the circle.

5. Through a given point to draw two straight lines at right angles to each other, such that the parts of them intercepted by a given circle may bear towards each other a given ratio.

Let c, c', be the lengths of the two intercepted chords, and ϕ the inclination of the former to the diameter passing through the given point; let r denote the radius of the circle, and a the distance of the given point from its centre. Let $c = \mu c'$, μ being a constant. Then

$$c = \mu \left(\frac{8r^2 - 4a^2}{1 + \mu^2}\right)^{\frac{1}{2}}, \qquad c' = \left(\frac{8r^2 - 4a^2}{1 + \mu^2}\right)^{\frac{1}{2}},$$

and
$$\sin \phi = \left\{\frac{r^2 (1 - \mu^2) + \mu^2 a^2}{(1 + \mu^2) a^2}\right\}^{\frac{1}{2}}.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 145, *troisième édition.*

6. To find the position of the centre of a circle which passes through a given point, touches one given straight line, and intercepts from another given straight line a chord of given length.

Let the summit of the angle included between the straight lines be taken as the pole, the straight line which touches the circle being the prime radius vector.

Let $f =$ the length of the given chord; let (β, ρ) be the coordinates of the given point, a the radius vector of the point of contact, c the perpendicular distance of the centre from the prime radius vector, α the angle between the two straight lines. Then a and c are to be determined from the two equations

$$2c\rho \sin \beta = \rho^2 + a^2 - 2a\rho \cos \beta,$$

$$\rho^2 (4a^2 + f^2) \sin^2 \beta = \{2a\rho \sin (\beta - \alpha) + (\rho^2 + a^2) \sin \alpha\}^2,$$

the latter being a biquadratic in a.

Bossut: *Géométrie*, p. 345.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 173, *troisième édition.*

SECTION VIII.

Polar Equations to Tangents and Chords.

1. If a constant moveable angle has its summit fixed at a point in the circumference of a circle, the chord, which it subtends in the circle, is always a tangent to another circle, concentric with the original one.

If $r = c$ be the equation to the circle; λ, μ, the angular co-ordinates of the chord in any position, and ε the constant angle; then, the perpendicular distance of the origin from the chord being $c \cos \frac{\lambda - \mu}{2}$, and $\frac{\lambda + \mu}{2}$ being the inclination of this distance to the prime radius vector, the equation to the chord will be

$$c \cos \frac{\lambda - \mu}{2} = r \cos \left(\theta - \frac{\lambda + \mu}{2}\right).$$

But from the geometry it is easily seen that

$$\lambda - \mu = 2(\pi - \varepsilon).$$

Hence the equation to the chord becomes

$$c \cos(\pi - \varepsilon) = r \cos \left(\theta - \frac{\lambda + \mu}{2}\right),$$

which is the equation to a series of straight lines, varying in position with the value of $\frac{\lambda + \mu}{2}$, the perpendicular distances of all of them from the origin being $c \cos(\pi - \varepsilon)$. The chord therefore always touches a concentric circle, the radius of which is $c \cos(\pi - \varepsilon)$.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 190.

2. To find the equation to the chord of a given circle, supposing it to subtend a right angle at the centre, and having given the position of one of its extremities.

Let α be the angular coordinate of the given extremity, and c be the radius of the circle. Then the equation required is

$$c = \sqrt{2} \,.\, r \,.\, \cos\left(\theta - \alpha \pm \frac{\pi}{4}\right).$$

SECTION IX.

Poles and Polars.

1. If any number of circles touch each other in one point, all their polars, which correspond to a common pole, will pass through a single point.

Let the point of contact of the circles be taken as the origin of coordinates, the common diameter through the origin as the axis of x, and the tangent as the axis of y.

The equations to any two of the circles will be

$$x^2 + y^2 = 2r'x,$$

$$x^2 + y^2 = 2r''x.$$

Let $(x_{\prime}, y_{\prime})$ be the common pole. Then the equations to the polars of these two circles will be

$$(x_{\prime} - r')\, x + y_{\prime} y = r' x_{\prime},$$

and $$(x_{\prime} - r'')\, x + y_{\prime} y = r'' x_{\prime}.$$

At the intersection of these two polars,

$$(r'' - r')\, x = -\, (r'' - r')\, x_{\prime},$$

$$x = -\, x_{\prime}, \qquad y = \frac{x_{\prime}^2}{y_{\prime}}.$$

Since the values of x and y are independent of the radii of the circles, the truth of the proposition is obvious.

2. To find the locus of the pole, when the polar of a given circle always passes through a given point.

Let the equation to the circle be

$$x^2 + y^2 = r^2,$$

the axis of x being so chosen as to pass through the given point.

Then $(x_{,}, y_{,})$, being the pole, the equation to the polar will be

$$x_{,}x + y_{,}y = r^2.$$

But $x = a$, $y = 0$, simultaneously, a being some constant quantity. Hence

$$x_{,} = \frac{r^2}{a},$$

the equation to the locus of the pole, which is therefore a line perpendicular to the line which passes through the centre of the circle and the given point.

3. A fixed circle is cut by a series of circles touching each other at one point; to find the locus of the pole of the fixed circle, the polar of which passes through the two points in which the fixed circle is cut by any one of the series of circles.

The equation to the fixed circle being

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2,$$

and that to any one of the series of circles

$$x^2 + y^2 = 2rx,$$

the equation to the required locus will be

$$(\gamma^2 - \alpha^2 + \beta^2)(y - \beta) + 2\gamma^2\beta + 2\alpha\beta(x - \alpha) = 0.$$

4. The equation to the polar of a circle, denoted by the equation

$$x^2 + y^2 = c^2,$$

being

$$\frac{x}{a} + \frac{y}{b} = 1,$$

to find the coordinates of the corresponding pole P, and, AB being the portion of the polar intercepted by the circle, to find the inclinations of AP, BP, to the axis of x.

The coordinates of P are $\frac{c^2}{a}$, $\frac{c^2}{b}$, and, t denoting the tangent of the inclination of AP or BP to the axis of x,

$$t^2\frac{a^2-c^2}{a^2}+\frac{2c^2}{ab}t+\frac{b^2-c^2}{b^2}=0.$$

SECTION X.

Radical Axes, Centres of Similitude, &c.

1. If three circles, described in one plane, intersect, two and two, their three chords of intersection pass through a single point.

Let the equations to the three circles be

$$(x-a)^2+(y-b)^2=c^2 \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$(x-a')^2+(y-b')^2=c'^2 \ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$(x-a'')^2+(y-b'')^2=c''^2 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Subtracting (2) from (1), we have, for the equation to the chord of intersection of the first two circles,

$$2(a'-a)x+2(b'-b)y+a^2-a'^2+b^2-b'^2=c^2-c'^2\ldots(4).$$

Similarly, for the intersections of (2), (3), and of (3), (1), we have

$$2(a''-a')x+2(b''-b')y+a'^2-a''^2+b'^2-b''^2=c'^2-c''^2\ldots(5),$$

and

$$2(a-a'')x+2(b-b'')y+a''^2-a^2+b''^2-b^2=c''^2-c^2\ldots(6).$$

The three lines, denoted by (4), (5), (6), as their equations indicate, pass through a single point.

The lines (4), (5), (6), have been called by Gaultier the *radical axes* of the pairs of circles (1, 2), (2, 3), (3, 1), respectively, and the point of intersection of (4), (5), (6), has been called by him the *radical centre* of the three circles. The terms *radical axes* are applied to the lines denoted by (4), (5), (6), and *radical centre* to their point of intersection, even when the circles do not cut each other.

Gaultier-de-Tours: *Journal de l'Ecole Polytechnique*, Cah. XVI.
Carnot: *Géométrie de Position*, p. 347.
Durrande: Gergonne, *Annales de Mathématiques*, tom. XI. p. 13.

2. The tangents drawn to two circles, from any point whatever in their radical axis, and terminated at their points of contact, are equal to each other.

Let the equations to the two circles be

$$(x-a')^2+(y-b')^2=c'^2,$$
$$(x-a'')^2+(y-b'')^2=c''^2.$$

The equation to the radical axis will be

$$2(a''-a')\,x+2(b''-b')\,y+a'^2-a''^2+b'^2-b''^2=c'^2-c''^2.$$

Suppose the origin to be so chosen as to lie in the radical axis; then

$$a'^2-a''^2+b'^2-b''^2=c'^2-c''^2,$$

or

$$a'^2+b'^2-c'^2=a''^2+b''^2-c''^2,$$

an algebraic expression of the truth of the proposition.

Durrande: *Gergonne, Annales de Mathématiques*, tom. VIII. p. 322.
Plucker: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 33.

3. To find the equations to the *internal* and *external polars of similitude* of any two circles.

The term *centre of similitude* was introduced by Monge. The *external centre of similitude* of two circles is the point of intersection of the two common external tangents, and the *internal centre* is the point of intersection of the two common internal tangents. Any straight line containing three of the *centres of similitude* of a system of three circles is called *an axis of similitude; an external axis*, when it contains the three *external centres of similitude*, and *an internal axis*, when it contains one *external* and two *internal centres of similitude.*

The *internal polar of similitude is the polar of the internal centre of similitude*, and the *external polar of similitude* is the *polar* of the *external centre of similitude.*

Let the equations to the two circles be

$$(x-a)^2+(y-b)^2=c^2\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$(x-a')^2+(y-b')^2=c'^2\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation to a tangent to (1), at a point (x, y), is

$$(X - a) . (x - a) + (Y - b) . (y - b) = c^2 \text{......} (3);$$

and that to (2), at a point (x', y'), is

$$(X - a') . (x' - a') + (Y - b') . (y' - b') = c'^2 \text{......} (4).$$

If the lines (3) and (4) are coincident, then, the coefficients of X, Y, having the same ratios in both equations, we must have

$$\lambda (x' - a') = x - a \text{....................} (5),$$

and

$$\lambda (y' - b') = y - b \text{....................} (6):$$

hence, from (3) and (4),

$$(a' - a) . (x - a) + (b' - b) . (y - b) = c^2 - \lambda c'^2 \text{...} (7).$$

But, from (1), (5), (6),

$$(x' - a')^2 + (y' - b')^2 = \frac{c^2}{\lambda^2},$$

and therefore, (x', y') being a point in (2),

$$c'^2 = \frac{c^2}{\lambda^2}, \quad \text{or } \lambda = \pm \frac{c}{c'}.$$

The equation (7) is therefore reduced to the form

$$(a' - a) (x - a) + (b' - b) (y - b) = c (c \mp c'),$$

and is equivalent to the two

$$(a' - a) (x - a) + (b' - b) (y - b) = c (c - c') \text{...} (8),$$

and

$$(a' - a) (x - a) + (b' - b) (y - b) = c (c + c') \text{...} (9).$$

Either of these equations, combined with (1), will give two pairs of values for the coordinates of contact: they will accordingly represent two chords of contact, perpendicular to the line joining the centres of the circles, viz. the two polars of similitude relatively to the circle (1).

Salmon: *Treatise on Conic Sections*, p. 100.

4. To find the coordinates of the centres of similitude of two circles.

The centres of similitude evidently lie in the line joining the centres of the two circles, the equation to which, if we adopt the circumstances and notation of the preceding problem, will be

$$\frac{Y-b}{b'-b}=\frac{X-a}{a'-a},$$

that to a common tangent to the circles at a point (x, y), being

$$(X-a)(x-a)+(Y-b)(y-b)=c^2.$$

From these two equations we have

$$(X-a)\{(a'-a)(x-a)+(b'-b)(y-b)\}=c^2(a'-a),$$

and therefore, if we suppose x, y, to be a point in the former of the polars of similitude,

$$X-a=c\frac{a'-a}{c-c'}, \quad X=\frac{a'c-c'a}{c-c'}, \quad Y=\frac{b'c-c'b}{c-c'}.$$

From these values of X, Y, we see that $X-a$, $Y-b$, have the same signs as $X-a'$, $Y-b'$, and that accordingly the point X, Y, lies in the prolongation of the line joining the centres of the circles. Hence these values of X, Y, belong to the *external centre of similitude*, and the equation (8) to the *external polar of similitude*.

If we suppose the point x, y, to be in the latter of the *polars of similitude*, we shall get for the coordinates of the *internal centre of similitude*

$$X=\frac{a'c+c'a}{c+c'}, \quad Y=\frac{b'c+c'b}{c+c'},$$

the equation (9) belonging accordingly to the *internal polar of similitude.*

Salmon: *Treatise on Conic Sections*, p. 101.

For further information respecting *centres* and *polars of similitude*, the student is referred to memoirs by Durrande; *Gergonne, Annales de Mathématiques*, tom. II. p. 1; tom. XVII. p. 285; and Plucker, *ib.* tom. XVIII. p. 29.

5. To find the locus of the radical centre of three circles, the radii of which are $r+\rho$, $r'+\rho$, $r''+\rho$, where ρ is a variable quantity, the centres of the three circles being fixed.

Let (a, b), (a', b'), (a'', b''), be the central coordinates of the three circles: then the required locus will be a straight line defined by the equation

$$\{r(a' - a'') + r'(a'' - a) + r''(a - a')\}\,x \\ + \{r(b' - b'') + r'(b'' - b) + r''(b - b')\}\,y = 0,$$

the position of the radical centre of the three circles, when $\rho = 0$, being taken as origin of coordinates.

Plucker: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 35.

6. To prove that the locus of the radical centre of the three circles

$$(x - a)^2 + (y - b)^2 = (r + \rho)^2,$$
$$(x - a')^2 + (y - b')^2 = (r' + \rho)^2,$$
$$(x - a'')^2 + (y - b'')^2 = (r'' + \rho)^2,$$

where ρ is a variable quantity, is at right angles to their external axis of similitude.

Plucker: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 36.

7. The four tangents, which are common to two circles which do not intersect, and are terminated at their points of respective contact, have their middle points on the radical axis of the two circles.

Gergonne: *Annales de Mathématiques*, tom. VIII. p. 323.

8. Let two circles be touched respectively by a single straight line AA' in A and A', and by a single circle $BB'C$ in B and B': if the straight line and the circle touch in the same manner the two circles, the point C of the concourse of AB and $A'B'$, will lie on the circumference of the circle $BB'C$ and on the radical axis of the two other circles.

Durrande: *Gergonne, Annales de Mathématiques*, tom. VIII. p. 324.

9. The external centres of similitude of three circles, taken successively two and two, all lie in one straight line; and each

of them is situated in a right line with two of the internal centres of similitude.

Fuss: *Nova Acta Academiæ Petropolitanæ*, tom. XIV. p. 140.
Durrande: *Gergonne, Annales de Mathématiques*, tom. XI. p. 8.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 165, *troisième édition.*

Fuss says, in relation to the former portion of this problem:

"Il y a déjà plusieurs années qu' un jeune François, employé alors au Corps Impérial des Cadets de Terre, me parla d' un Théoréme de Géométrie qui, dans le tems qu' il étoit encore à Paris à l' Ecole Royale militaire, avoit eu quelque célébrité et qu' on avoit prétendu tenir de feu Mr. d'Alembert. Je lui en donnai une démonstration, dont j' ai retrouvé depuis peu le brouillon en fouillant mes papiers. En relisant cette démonstration j' ai vu que la belle propriété qui en fait le sujet, peut conduire à d' autres non moins remarquables. En rassemblant mes idées sur cette matière il en est résulté le petit Mémoire que j'ai l' honneur de présenter ici à l' Académie pour la collection des Mémoires traduits en Russe, qu' elle se propose de publier, ou bien pour les Actes mêmes, si elle le juge digne de cet honneur. Il y fera sans doute plaisir à plus d' un amateur de la Géométrie, et peut-être même à quelque Géométre de profession."

10. The three circles

$$(x-a)^2+(y-b)^2=r^2, \qquad (x-a')^2+(y-b')^2=r'^2,$$
$$(x-a'')^2+(y-b'')^2=r''^2,$$

are touched by a fourth circle: to prove that the point, in which the third circle is touched, lies in the line of which the equation is

$$\frac{(a-a'').(x-a'')+(b-b'').(y-b'')+(r-r'')\,r''}{(a-a'')^2+(b-b'')^2-(r-r'')^2}$$
$$=\frac{(a'-a'').(x-a'')+(b'-b'').(y-b'')+(r'-r'')\,r''}{(a'-a'')^2+(b'-b'')^2-(r'-r'')^2}.$$

For full information on the subject of the contact of a fourth circle with any three proposed circles, the reader is referred to Euler,[1] Fuss,[2] Schubert,[3] Gergonne,[4] Puissant,[5] Montucla,[6] Hearn,[7] Salmon.[8]

SECTION XI.

Inscribed and Circumscribed Polygons.

1. $ABCP$ and $A'B'C'P'$ are two concentric circles, ABC and $A'B'C'$ any two equilateral triangles inscribed in them. If P, P', be any two points in the circumferences of these circles, to shew that

$$(A'P)^2 + (B'P)^2 + (C'P)^2 = (AP')^2 + (BP')^2 + (CP')^2.$$

Let the equations to the two circles be

$$x^2 + y^2 = r^2, \qquad x^2 + y^2 = r'^2.$$

Let (x, y) be the coordinates of P, and (x_1', y_1'), (x_2', y_2'), (x_3', y_3'), those of A', B', C', respectively. Then

$$\begin{aligned}
&(A'P)^2 + (B'P)^2 + (C'P)^2 \\
&\quad = (x_1'-x)^2 + (y_1'-y)^2 + (x_2'-x)^2 + (y_2'-y)^2 + (x_3'-x)^2 + (y_3'-y)^2 \\
&\quad = x_1'^2 + y_1'^2 + x_2'^2 + y_2'^2 + x_3'^2 + y_3'^2 + 3(x^2 + y^2) \\
&\qquad\qquad - 2x(x_1' + x_2' + x_3') - 2y(y_1' + y_2' + y_3') \\
&\quad = 3(r'^2 + r^2) - 2x(x_1' + x_2' + x_3') - 2y(y_1' + y_2' + y_3').
\end{aligned}$$

But, $r'\sqrt{3}$ being the side of the triangle in the circle $A'B'C'P'$,

$$3r'^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 = 2r'^2 - 2(x_1'x_2' + y_1'y_2'),$$

$$2(x_1'x_2' + y_1'y_2') = -r'^2 = -(x_1'^2 + y_1'^2).$$

Similarly $\quad 2(x_2'x_3' + y_2'y_3') = -(x_2'^2 + y_2'^2),$

and $\quad 2(x_3'x_1' + y_3'y_1') = -(x_3'^2 + y_3'^2).$

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[1] *Nova Acta Academiæ Petropolitanæ*, tom. VI. p. 95.

[2] *Ibid.* tom. VI. p. 102.

[3] *Ibid.* tom. X. p. 77.

[4] *Annales de Mathématiques*, tom. VII. p. 289.

[5] *Recueil de diverses propositions de Géométrie*, p. 176, *troisième édition*.

[6] *Histoire des Mathématiques*, tom. I. p. 263.

[7] *Researches on Curves of the Second Order*, p. 22.

[8] *Treatise on Conic Sections*, p. 104, &c.

Adding together these three equations, we have

$$(x_1' + x_2' + x_3')^2 + (y_1' + y_2' + y_3')^2 = 0,$$

and therefore

$$x_1' + x_2' + x_3' = 0, \qquad y_1' + y_2' + y_3' = 0.$$

Hence $\quad (A'P)^2 + (B'P)^2 + (C'P)^2 = 3\,(r^2 + r'^2).$

Also, by symmetry,

$$(AP')^2 + (BP')^2 + (CP')^2 = 3\,(r^2 + r'^2).$$

2. Any quadrilateral being described about a circle, to prove that the line, which joins the middle points of the diagonals, passes through the centre of the circle.

Let the equation to the circle be

$$x^2 + y^2 = c^2:$$

this equation is equivalent to the two following,

$$x + y\sqrt{(-1)} = c\alpha, \qquad x - y\sqrt{(-1)} = \frac{c}{\alpha},$$

where α is arbitrary: whence we have

$$x = \tfrac{1}{2}c\left(\alpha + \frac{1}{\alpha}\right), \qquad y = \tfrac{1}{2}\frac{c}{\sqrt{(-1)}}\left(\alpha - \frac{1}{\alpha}\right).$$

The equation

$$x\left(\alpha + \frac{1}{\alpha}\right) + \frac{y}{\sqrt{(-1)}}\left(\alpha - \frac{1}{\alpha}\right) = 2c \;\ldots\ldots\ldots\; (1)$$

represents therefore a tangent to the circle, which we will suppose to coincide with a side of the quadrilateral. The equation

$$x\left(\alpha' + \frac{1}{\alpha'}\right) + \frac{y}{\sqrt{(-1)}}\left(\alpha' - \frac{1}{\alpha'}\right) = 2c \ldots\ldots\ldots\ldots (2)$$

may be taken to represent the next side.

At the intersection $x_{1,2}$, $y_{1,2}$, of these two lines, as may be easily shewn,

$$x_{1,2} = c\,\frac{1 + \alpha\alpha'}{\alpha + \alpha'}, \qquad y_{1,2} = c\sqrt{(-1)}\,\frac{1 - \alpha\alpha'}{\alpha + \alpha'}.$$

Similarly, at the intersection of the other two sides,

$$x_{3,4} = c\,\frac{1+\alpha''\alpha'''}{\alpha''+\alpha'''}, \qquad y_{3,4} = c\sqrt{(-1)}\,\frac{1-\alpha''\alpha'''}{\alpha''+\alpha'''}.$$

If X, Y, denote the coordinates of the middle point of the diagonal through these two intersections, we have

$$X = \tfrac{1}{2}c\left(\frac{1+\alpha\alpha'}{\alpha+\alpha'} + \frac{1+\alpha''\alpha'''}{\alpha''+\alpha'''}\right),$$

$$Y = \tfrac{1}{2}c\sqrt{(-1)}\left(\frac{1-\alpha\alpha'}{\alpha+\alpha'} + \frac{1-\alpha''\alpha'''}{\alpha''+\alpha'''}\right);$$

whence, performing obvious simplifications,

$$\frac{Y}{X} = \sqrt{(-1)}\,.\,\frac{\alpha(1-\alpha'\alpha'')+\alpha'(1-\alpha''\alpha''')+\alpha''(1-\alpha'''\alpha)+\alpha'''(1-\alpha\alpha')}{\alpha(1+\alpha'\alpha'')+\alpha'(1+\alpha''\alpha''')+\alpha''(1+\alpha'''\alpha)+\alpha'''(1+\alpha\alpha')}.$$

This result, being symmetrical in regard to α, α', α'', α''', shews that in the case of the other diagonal, we shall have the same ratio for the coordinates X', Y', of its middle point. Hence

$$\frac{Y'}{X'} = \frac{Y}{X},$$

which shews that the two middle points are in a line through the centre of the circle.

Durrande: *Gergonne, Annales de Mathématiques*, tom. XIV. p. 309.

3. To prove that the equation to a circle, circumscribed about a triangle, the equations to the sides of which are

$$u = x\cos\alpha + y\sin\alpha - p,$$

$$v = x\cos\beta + y\sin\beta - q,$$

$$w = x\cos\gamma + y\sin\gamma - r,$$

is

$$vw\sin A + wu\sin B + uv\sin C = 0,$$

A, B, C, being the angles between the sides (v, w), (w, u), (u, v), respectively.

Salmon: *Treatise on Conic Sections*, p. 92.

4. If two triangles are the one inscribed in a circle and the other circumscribed about it; so that the summits of the

inscribed are the points of contact of the circumscribed triangle; to prove that, (1), the points of concourse of the directions of the opposite sides of the two triangles lie in a single straight line, (2), that the lines joining their opposite summits meet all three in a single point, and, (3), that this point is the pole of the aforesaid straight line.

Durrande: *Gergonne, Annales de Mathématiques*, tom. XIV. p. 45.
Salmon: *Treatise on Conic Sections*, p. 94.

5. The feet of the perpendiculars let fall, upon the directions of the sides of any triangle, from any point in the circumference of the circumscribed circle, all lie in one straight line.

Servois, Gergonne: *Gergonne, Annales de Mathématiques*, tom. IV. p. 251.
Durrande: *Ibid.* tom. VII. p. 253.
Querret: *Ibid.* tom. XIV. p. 285.
Salmon: *Treatise on Conic Sections*, p. 94.

6. The base OA of a triangle POA is given, and the triangle is such that a circle of given radius may always be inscribed in it: to find the locus of P.

If $OP=r$, $\angle POA=\theta$, $OA=a$, $c=$ the radius of the inscribed circle, the equation to the locus of P will be

$$\sin\theta\,.\,(a\sin\theta-2c)\,r=2c\,(a\sin\theta-c\cos\theta-c).$$

Manderlier: *Quetelet, Corr. Math. et Phys.*, tom. II. p. 320.

7. To find the radius of a circle inscribed in a triangle the equations to the sides of which are

$$x\cos\alpha+y\sin\alpha=\delta,\qquad x\cos\alpha'+y\sin\alpha'=\delta',$$
$$x\cos\alpha''+y\sin\alpha''=\delta''.$$

The radius is equal to

$$\frac{\delta\sin(\alpha'-\alpha'')+\delta'\sin(\alpha''-\alpha)+\delta''\sin(\alpha-\alpha')}{4\sin\frac{\alpha'-\alpha''}{2}\,.\,\sin\frac{\alpha''-\alpha}{2}\,.\,\sin\frac{\alpha-\alpha'}{2}}.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 119.

8. The equations to three straight lines are

$$u = 0, \quad u_1 = 0, \quad u_2 = 0,$$

$x \sin\theta - y \cos\theta + c = 0$, being taken as the type of the equation to a straight line: to prove that the equations of the four circles, to each of which these lines are tangents, are

$$u^{\frac{1}{2}} \sin\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \sin\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \sin\frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \sin\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \cos\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \cos\frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \cos\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \sin\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \cos\frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \cos\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \cos\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \sin\frac{\theta_1 - \theta}{2} = 0.$$

Salmon: *Treatise on Conic Sections*, p. 94.

9. Having given the equations to the three sides of a triangle, to find the equations to the lines joining the angles of the triangle with the points in which the escribed circles touch the opposite sides.

The notation of the preceding problem being retained, the equations to the three lines will be

$$u_1 \cos^2\frac{\theta - \theta_2}{2} = u_2 \cos^2\frac{\theta_1 - \theta}{2}, \quad u_2 \cos^2\frac{\theta_1 - \theta}{2} = u \cos^2\frac{\theta_2 - \theta_1}{2},$$

$$u \cos^2\frac{\theta_2 - \theta_1}{2} = u_1 \cos^2\frac{\theta - \theta_2}{2}.$$

These equations shew that the three lines pass through a single point.

10. If any number of quadrilaterals, inscribed in a given circle, have a common side, to prove that the lines which, in these quadrilaterals, join the intersection of the diagonals and the point of concourse of the sides adjacent to the common side,

will all meet in a single point of the perpendicular on the middle point of the common side.

Durrande: *Gergonne, Annales de Mathématiques*, tom. XIII. p. 308.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 137, *troisième edition.*

SECTION XII.

Circular Loci.

1. The distances of a point P from two fixed points A, A', are r, r'. To find the locus of P, supposing that $nr^2 + n'r'^2$ is equal to a constant quantity c^2, where n, n', are given quantities.

Let (a, b), (a', b'), be the coordinates of A, A', respectively, (x, y) being those of P. Then

$$n\{(x-a)^2 + (y-b)^2\} + n'\{(x-a')^2 + (y-b')^2\} = c^2,$$

$$(n+n')(x^2+y^2) - 2x(na+n'a') - 2y(nb+n'b') + n(a^2+b^2) + n'(a'^2+b'^2) = c^2,$$

$$x^2 - 2\frac{na+n'a'}{n+n'}\cdot x + y^2 - 2\frac{nb+n'b'}{n+n'}\cdot y = \frac{c^2 - n(a^2+b^2) - n'(a'^2+b'^2)}{n+n'},$$

$$\left(x - \frac{na+n'a'}{n+n'}\right)^2 + \left(y - \frac{nb+n'b'}{n+n'}\right)^2 = \frac{c^2}{n+n'} - \frac{nn'}{(n+n')^2}\cdot\{(a-a')^2 + (b-b')^2\},$$

the equation to the required locus, which is therefore a circle, its radius being equal to

$$\frac{1}{n+n'}\left[(n+n')c^2 - nn'\{(a-a')^2 + (b-b')^2\}\right]^{\frac{1}{2}},$$

and central coordinates to

$$\frac{na+n'a'}{n+n'}, \quad \frac{nb+n'b'}{n+n'}.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 131.

2. To find the locus of a point such that the sum of the squares of its distances from any number of proposed points may have a given value.

Let there be n proposed points (α_1, β_1), (α_2, β_2), (α_3, β_3), ... (α_n, β_n), and let (x, y) be the coordinates of the moveable point. Also let c^2 be the constant value of the sum of the squares. Then, the axes being rectangular, we have

$$\begin{aligned} &(x-\alpha_1)^2+(y-\beta_1)^2 \\ &+(x-\alpha_2)^2+(y-\beta_2)^2 \\ &+(x-\alpha_3)^2+(y-\beta_3)^2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+(x-\alpha_n)^2+(y-\beta_n)^2=c^2, \end{aligned}$$

or

$$x^2+y^2-\frac{2}{n}(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n)\,x-\frac{2}{n}(\beta_1+\beta_2+\beta_3+\ldots+\beta_n)\,y$$
$$=\frac{1}{n}\{c^2-(\alpha_1^2+\alpha_2^2+\alpha_3^2+\ldots+\alpha_n^2)-(\beta_1^2+\beta_2^2+\beta_3^2+\ldots+\beta_n^2)\},$$

the equation to the required locus, which is therefore a circle the centre of which has for its coordinates

$$\frac{1}{n}(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n), \qquad \frac{1}{n}(\beta_1+\beta_2+\beta_3+\ldots+\beta_n).$$

Apollonii Pergœi Locorum Planorum Libri II. *Restituti a Roberto Simson*, p. 159.

Montucla: *Histoire des Mathématiques*, tom. I. p. 284, 2^e *édit.*

Puissant: *Recueil de diverses propositions de Géométrie*, p. 194, *troisième édition.*

Garnier: *Géométrie Analytique*, p. 183, *deuxième édition.*

3. There are $2n$ given straight lines, which make with another given straight line angles $\alpha, \beta, \gamma, \ldots\ldots$ A point P is taken such that the sum of the squares of the perpendiculars drawn from it upon these $2n$ lines is constant. To prove that the necessary and sufficient condition, that the locus of P may be a circle, is

$$n+\cos 2(\alpha-\beta)+\cos 2(\beta-\gamma)+\cos 2(\alpha-\gamma)+\ldots=0.$$

Take the fixed line of reference as the axis of y, and any line at right angles to it as that of x. Then the equations to the $2n$ lines will be of the forms

$$\begin{aligned} x\cos\alpha + y\sin\alpha &= a,\\ x\cos\beta + y\sin\beta &= b,\\ x\cos\gamma + y\sin\gamma &= c,\\ &\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots \end{aligned}$$

The condition of the problem, Δ denoting the constant value of the sum of the squares, gives us

$$(x\cos\alpha + y\sin\alpha - a)^2 + (x\cos\beta + y\sin\beta - b)^2$$
$$+ (x\cos\gamma + y\sin\gamma - c)^2 + \ldots\ldots = \Delta^2.$$

In order that this equation may represent a circle, it is sufficient and necessary that the coefficients of x^2 and y^2 be equal, and that the coefficient of xy be zero: hence

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \ldots\ldots = 0,$$

and $$\sin 2\alpha + \sin 2\beta + \sin 2\gamma + \ldots\ldots = 0.$$

These two equations are equivalent to the single equation

$$(\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \ldots)^2 + (\sin 2\alpha + \sin 2\beta + \sin 2\gamma + \ldots)^2 = 0,$$

or, squaring the two members, simplifying and dividing by 2,

$$n + \cos 2(\alpha - \beta) + \cos 2(\beta - \gamma) + \cos 2(\alpha - \gamma) + \ldots = 0.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 142.

4. To find the locus of the intersection of a perpendicular from a fixed point upon a line passing through another fixed point.

Let the former fixed point be taken as the origin of polar coordinates: then, p denoting the perpendicular and ϕ its inclination to the prime radius vector, the polar equation to the line mentioned in the problem will be

$$p = r\cos(\theta - \phi).$$

Let r', θ', be the given point: then

$$p = r' \cos(\theta' - \phi).$$

This result shews that p, ϕ, are the polar coordinates of any point in a circle of which r' is the diameter, this circle being therefore the required locus.

5. To determine the geometrical locus of the point which divides into two parts, in a constant ratio, the straight line drawn from a given point to any point in the circumference of a given circle.

Let the circle be referred to two diameters as axes of x and y, the axis of x being so chosen as to pass through the given point. The equation to the circle will be, c denoting its radius,

$$x^2 + y^2 = c^2 \text{ .... } (1).$$

Let a be the abscissa of the given point C, and x', y', the coordinates of the dividing point P'. Let CP' meet the circle in P, and let $CP' = n.CP$, n being, by the condition of the problem, constant.

Then it is easily seen, by similar triangles, x, y, being the coordinates of P, that

$$a - x' = n(a - x),$$

or

$$nx = x' + (n - 1)a,$$

and

$$ny = y'.$$

But, from (1), $\quad (nx)^2 + (ny)^2 = (nc)^2$:

hence we have, for the equation to the required locus,

$$\{x' + (n - 1)a\}^2 + y'^2 = (nc)^2,$$

which is therefore a circle, the radius of which is nc, its centre lying in the axis of x at a distance $(1 - n)a$ from the origin.

Cor. Suppose the equation to the locus of P' to be given, and let it be $\quad (x' + h)^2 + y'^2 = c'^2.$

Then, for the determination of the position of C and of the magnitude of n, we have

$$(n-1)\,a = h, \quad c' = nc,$$

and therefore

$$n = \frac{c'}{c}, \qquad a = \frac{ch}{c' - c}.$$

Remark. Thus, two circles being given, we may find the position of such a point and also determine such a ratio, that, if we draw through the point any straight line to cut the circumferences of the two circles, the ratio between the parts of this straight line, comprised between this point and the circumferences, may be equal to the ratio.

The proposition which constitutes the subject of this remark belongs to a class of propositions, to which (according to Simson*) the ancients gave the name of *porisms.* While a locus is determined by certain independent conditions; on the other hand, when a locus is proposed, it determines certain relations among the conditions by which it would have been itself determined; so that two or more of these conditions are simultaneously determined by the proposed locus. Thus, the locus of the point P', which is the object of the present problem, being proposed, the position of the point C and the ratio of CP to CP' are conjointly determined.

Playfair† has taken a different view of the nature of a *porism;* he has arrived at the conclusion that the ancients gave the name of *porisms* to those propositions which affirm the possibility of finding conditions such as to render a particular problem indeterminate or susceptible of an infinite number of solutions.

Simson's definition of a porism is the following: "Porisma est Propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quæ ad ea quæ data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in Propositione descriptam;" or, in Playfair's translation of these words, "a Porism is a proposition, in which it is proposed to demonstrate, that one or more things are given, between which and every one of innumerable other things, not given but assumed according to a given law, a certain relation, described in the proposition, is to be shewn to take place."

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* *De Porismatibus Tractatus; Opera quædam reliqua.* 1776, Glasgow.

† *Transactions of the Royal Society of Edinburgh*, vol. III.

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 39.

Lamé: *Examen des différentes Méthodes employées pour résoudre les Problèmes de Géométrie*, p. 22.

6. To find on the circumference of a given circle a point such that the sum of the squares of its distances from two given points shall be equal to a given area.

Let the equation to the given circle be

$$x^2 + y^2 = c^2 \text{.......................} (1);$$

and let (a, b), (a', b'), be the coordinates of the given points.

Then, m^2 representing the given area, we have, by the condition of the problem,

$$(x-a)^2 + (y-b)^2 + (x-a')^2 + (y-b')^2 = m^2,$$

or $x^2 + y^2 - (a+a')x - (b+b')y = \frac{1}{2}(m^2 - a^2 - b^2 - a'^2 - b'^2) \dots (2).$

From (1) and (2) we have

$$(a+a')x + (b+b')y = \tfrac{1}{2}(2c^2 - m^2 + a^2 + b^2 + a'^2 + b'^2) \dots (3).$$

Thus the required point will be an intersection of the circle (1) with the chord (2).

Cor. Suppose that

$$a + a' = 0, \quad b + b' = 0,$$

and $$2c^2 - m^2 + a^2 + b^2 + a'^2 + b'^2 = 0,$$

or $$a^2 + b^2 + c^2 = \tfrac{1}{2}m^2 = a'^2 + b'^2 + c^2.$$

Then the equation (2) becomes an identical equation, and the problem becomes indeterminate; any point whatever in (1) satisfying the conditions of the problem; the required point being thus replaced by a circular locus of appropriate points.

This problem affords an exemplification of Playfair's interpretation of the ancient signification of the word *porism*.

Playfair: *Transactions of the Royal Society of Edinburgh*, vol. III.
Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 204.

7. A and B being two fixed points and P a point such that $AP = \mu.BP$, to find the locus of P.

The locus is a circle, the centre of which is in the line AB at a distance from A equal to

$$\frac{\mu^2 a}{\mu^2 - 1},$$

and of which the radius is equal to

$$\frac{\mu a}{\mu^2 - 1}.$$

Newton: *Arithmetica Universalis*, prob. XXVI.
L'Hospital: *Traité Analytique des Sections Coniques*, p. 252.
Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Agébrique*, p. 129.

8. To find the locus of a point such that the square of its distance from a point (a, b) is equal to the rectangle contained by its distance from a line

$$x \cos \alpha + y \sin \alpha = \delta,$$

and a given line $2c$.

The required locus is a circle defined by the equation

$$\{x-(a+c\cos\alpha)\}^2+\{y-(b+c\sin\alpha)\}^2=2c\{\delta+(a\cos\alpha+b\sin\alpha)+\tfrac{1}{2}c\}.$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 131.

9. To find the locus of the intersection of two straight lines, which pass through two given points, and are inclined to each other at a given angle.

Let the axes be so chosen that the one point coincides with the origin and that $(a, 0)$ are the coordinates of the other point. Then, c denoting the tangent of the constant angle included between the two lines, the locus will be a circle defined by the equation

$$x^2 + y^2 = \frac{a}{c}(cx + y).$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 199, *troisième édition*.

10. AB, AC, are two straight lines given in position; DE is a straight line of given length terminating in them at D, E, respectively; from D, E, are drawn perpendiculars to AB, AC, respectively, intersecting each other in P. To find the locus of P.

Let AB, AC, produced indefinitely, be taken as axes of x, y. Let $DE = c$, $\angle BAC = \omega$. Then the required locus will be a circle defined by the equation

$$x^2 + y^2 + 2xy \cos \omega = \frac{c^2}{\sin^2 \omega}.$$

Leybourn: *Mathematical Repository*, *New Series*, vol. I. p. 115.

11. Given one side of a triangle and the opposite angle, to find the locus of the intersection of the bisectors of the other two angles.

The middle point of the given side being taken as the origin, and a perpendicular to it through this point as the axis of y, the locus required will be a circle defined by the equation

$$x^2 + y^2 + 6ay \cot \beta = 9a^2,$$

$2a$ being the given side and β the given angle.

Lardner: *Algebraic Geometry*, p. 110.

12. Straight lines are drawn from the extremities of a given diameter of a circle to the extremities of a chord which always subtends a given angle at the centre: to find the locus of the intersection of the straight lines.

$x^2 + y^2 + 2ay \tan \frac{\beta}{2} = a^2$

Let α = the given angle, a = the radius of the circle. Let the centre of the circle be taken as origin of rectangular coordinates, and the given diameter as axis of x.

Then the required locus will be a circle represented by the equation

$$x^2 + y^2 \pm 2ay \tan\frac{\alpha}{2} = a^2,$$

the + or − sign being taken accordingly as the diameter and chord are joined towards the same or opposite parts.

13. To find the locus of a point, such that, if lines be drawn from it through the three summits of a triangle, and through each summit be drawn a perpendicular to the line passing through it, the three perpendiculars shall always pass through some one point.

Let one summit of the triangle be taken as the origin of rectangular coordinates; and let the coordinates of the other two summits be (a, b) and (a', b'). Let c, c', be the lengths of the two sides which meet in the origin, and α the angle between them.

Then the required locus will be a circle, defined by the equation

$$cc'(x^2 + y^2)\sin\alpha + c'^2(bx - ay)$$
$$- c^2(b'x - a'y) = 0,$$

being therefore the circumscribed circle of the triangle.

Stein: *Gergonne, Annales de Mathématiques*, tom. XV. p. 73.

14. A straight line is drawn through a given point C within a circle, to cut it in points P, P'. If a point p be taken in this straight line, such that

$$(Cp)^2 = CP.CP',$$

to find the locus of p.

If a be the distance of C from the centre, and r be the radius of the given circle, the locus of p will be a circle of which C is the centre, and of which the radius is equal to $(r^2 - a^2)^{\frac{1}{2}}$.

15. A straight line, drawn through a point O, cuts two given circles, the centres of which are C, C', in P, Q, and P', Q', respectively. To find the locus of O, having given that

$$OP.OQ = \mu.OP'.OQ'.$$

The equations to the two circles being of the forms

$$(x-a)^2 + y^2 = c^2, \quad (x-a')^2 + y^2 = c'^2,$$

the locus of O will be a circle, defined by the equation

$$(x-a)^2 + y^2 - c^2 = \mu\{(x-a')^2 + y^2 - c'^2\}.$$

Cor. 1. The centre of this circle will lie in CC' or $C'C$ produced accordingly as μ is $<$ or $>$ 1.

Cor. 2. If $\mu = 1$, the locus of O becomes a straight line.

16. From a point are drawn perpendiculars upon the sides of a regular polygon. To find the locus of the point, supposing the sum of the squares of the perpendiculars to be equal to a given area.

Let the centre of the polygon be taken as origin of rectangular coordinates; let n be the number of the sides of the polygon, c the distance of its centre from each side, and S the given area.

Then the required locus is a circle represented by the equation

$$x^2 + y^2 = 2\left(\frac{S}{n} - c^2\right).$$

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 135.
Garnier: *Géométrie Analytique*, p. 187.

17. The equations to the four sides of a quadrilateral taken successively are

$$u = x\cos\alpha + y\sin\alpha - \delta = 0, \quad u' = x\cos\alpha' + y\sin\alpha' - \delta' = 0,$$
$$u'' = x\cos\alpha'' + y\sin\alpha'' - \delta'' = 0, \quad u''' = x\cos\alpha''' + y\sin\alpha''' - \delta''' = 0:$$

the rectangle between the distances of a point P from the first and third of these lines is to that between its distances from the second and fourth, as n to n'. To find the conditions that the locus of P may be a circle.

The conditions are $n = n'$,

and $\alpha + \alpha'' = \alpha' + \alpha'''$, or $\alpha + \alpha'' = 2\pi - (\alpha' + \alpha''')$.

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 144.

18. To find the locus of a point in the plane of a triangle, such that, if perpendiculars be drawn from it upon the directions of its sides, the area of the triangle, formed by straight lines joining the feet of the three perpendiculars, may be constant.

If the equations to the three sides of the triangle be

$x \cos\alpha + y \sin\alpha = \delta$, $x \cos\alpha' + y \sin\alpha' = \delta'$, $x \cos\alpha'' + y \sin\alpha'' = \delta''$,

and k^2 be the constant area, the required locus will be two circles denoted by the double equation

$$\begin{aligned}
\pm 2k^2 = (x^2+y^2) \sin\alpha \sin\alpha' \sin\alpha'' &\left\{\frac{\sin(\alpha-\alpha')}{\sin\alpha''} + \frac{\sin(\alpha'-\alpha'')}{\sin\alpha} + \frac{\sin(\alpha''-\alpha)}{\sin\alpha'}\right\} \\
&+ \delta \sin(\alpha'-\alpha'') \{x \cos(\alpha'+\alpha''-\alpha) + y \sin(\alpha'+\alpha''-\alpha)\} \\
&+ \delta' \sin(\alpha''-\alpha) \{x \cos(\alpha''+\alpha-\alpha') + y \sin(\alpha''+\alpha-\alpha')\} \\
&+ \delta'' \sin(\alpha-\alpha') \{x \cos(\alpha+\alpha'-\alpha'') + y \sin(\alpha+\alpha'-\alpha'')\} \\
&+ \delta\delta'\delta'' \left\{\frac{\sin(\alpha-\alpha')}{\delta''} + \frac{\sin(\alpha'-\alpha'')}{\delta} + \frac{\sin(\alpha''-\alpha)}{\delta'}\right\}.
\end{aligned}$$

Querret: *Gergonne, Annales de Mathématiques*, tom. XIV. p. 280.
Sturm: *Ibid.*, tom. XIV. p. 286.

PARABOLA.

Section I.

Referred to the Axis and its Tangent. Ordinates.

1. A parabola, of which the equation is

$$y^2 = lx,$$

is cut in four points by the circle

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2:$$

to find the product of the distances of the four points of intersection from the axis of the parabola.

Eliminating x between the equations to the two curves, we get

$$(y^2 - l\alpha)^2 + l^2 (y - \beta)^2 = l^2\gamma^2,$$

or

$$y^4 + \ldots\ldots + l^2 (\alpha^2 + \beta^2 - \gamma^2) = 0.$$

Whence, by the theory of equations, the required product is equal to

$$l^2 (\alpha^2 + \beta^2 - \gamma^2),$$

or, a being the distance between the centre of the circle and the vertex of the parabola, to

$$l^2 (a^2 - \gamma^2).$$

2. Two ordinates of a parabola meet its axis in points equidistant from the focus. If the vertex be joined with the point, where one of the ordinates meets the parabola, to find the equation to the locus of the point where this line intersects the other ordinate.

The equation to the parabola being

$$y^2 = 4mx,$$

let the abscissæ of the two points be x and x'.

Then, by the hypothesis,

$$x + x' = 2m \text{ .... } (1).$$

The equation to the line joining the former point and the vertex is

$$y_{,} = \frac{y}{x} . x_{,}$$

$$= 2\left(\frac{m}{x}\right)^{\frac{1}{2}} x_{,} \text{ .... } (2),$$

and the equation to the ordinate of the other point is

$$x_{,} = x' \text{ .... } (3).$$

At the intersection of (2) and (3),

$$x + x' = 4m\,\frac{x_{,}^2}{y_{,}^2} + x_{,},$$

and therefore, by (1),

$$(2m - x_{,})\, y_{,}^2 = 4m x_{,}^2,$$

the equation connecting the coordinates of every point of the curve formed by the intersection of the line through the vertex and the other ordinate, that is, the equation to the required locus.

3. The rectangle contained between two ordinates $y_{,}$, $y_{,,}$, of a parabola $y^2 = 4mx$, is equal to m^2, the distance between these ordinates being equal to m: to find the magnitudes of $y_{,}$ and $y_{,,}$.

$$y_{,} = m(\sqrt{5} - 2)^{\frac{1}{2}}, \qquad y_{,,} = m(\sqrt{5} + 2)^{\frac{1}{2}}.$$

4. To prove that the area of a triangle inscribed in a parabola is equal to

$$\frac{1}{8m}(y' \sim y'').(y'' \sim y''').(y''' \sim y'),$$

where y', y'', y''', are the ordinates of the vertices of the triangle, $y^2 = 4mx$ being the equation to the curve.

5. The abscissa and double ordinate of a segment of a common parabola are a and b, and the diameters of its circumscribed and inscribed circles D and d; to prove that

$$D + d = a + b.$$

6. If a parabola intersects a circle in four points, to prove that the ordinates of the points of intersection, which lie on one side of the axis of the parabola, are together equal to the sum of the ordinates of the points of intersection, which lie on the other side of the axis.

De La Hire: *Sectiones Conicæ*, lib. v. prop. 29.

SECTION II.

Referred to the Axis and its Tangent. *Tangents.*

1. Two tangents are drawn to a parabola; the one touches it at a point (a, b), the other at a point (a', b'). To find the point in which the two tangents intersect.

If $4m$ denote the latus-rectum, the equations to the tangents will be

$$by = 2m(x + a),$$
$$b'y = 2m(x + a').$$

At the intersection of these two lines, eliminating x, and observing that b^2, b'^2, are respectively equal to $4ma$, $4ma'$, we have

$$(b - b')y = 2m(a - a')$$
$$= \tfrac{1}{2}(b^2 - b'^2),$$

whence

$$y = \tfrac{1}{2}(b + b').$$

Also, eliminating y, we see that

$$x = \frac{ab' - a'b}{b - b'}.$$

Thus the coordinates of the required point are

$$\frac{ab' - a'b}{b - b'} \text{ and } \frac{b + b'}{2}.$$

2. In the parabola, of which the equation is

$$y^2 = 4mx,$$

two tangents are drawn at points of which the abscissæ are in the ratio of $1 : \mu$; to find the equation to the locus of their intersection.

If (x', y'), (x'', y''), be the two points of contact, the equations to the two tangents will be

$$yy' = 2mx + \tfrac{1}{2}y'^2,$$

$$yy'' = 2mx + \tfrac{1}{2}y''^2.$$

But $\quad y'^2 = 4mx', \quad y''^2 = 4mx'' = 4\mu mx',$

and therefore $\quad y'' = \pm\, \mu^{\frac{1}{2}} y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$

At the intersection of the two tangents, we have, from their equations,

$$y\,(y'' - y') = \tfrac{1}{2}(y''^2 - y'^2),$$

$$y = \tfrac{1}{2}(y'' + y')$$

$$= \tfrac{1}{2}(1 \pm \mu^{\frac{1}{2}})\,y', \text{ by (1).}$$

Hence, from the former of the equations to the tangents, we see that

$$2mx = y'\,(y - \tfrac{1}{2}y') = \frac{2y}{1 \pm \mu^{\frac{1}{2}}} \cdot \left(y - \frac{y}{1 \pm \mu^{\frac{1}{2}}}\right),$$

$$y^2 = (1 \pm \mu^{\frac{1}{2}})^2 \cdot \frac{mx}{\pm\, \mu^{\frac{1}{2}}};$$

the appropriate equation to the locus being, by virtue of (1),

$$y^2 = (\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}})^2\, mx,$$

or

$$y^2 = -\,(\mu^{\frac{1}{4}} - \mu^{-\frac{1}{4}})^2\, mx,$$

accordingly as the points of contact are on the same or on opposite sides of the axis of the parabola.

3. To find the area of the triangle included between the tangents to parabolæ

$$y^2 = 4mx, \qquad y^2 = 4\mu x,$$

at points the common abscissa of which is a, and the portion of the ordinate intercepted between the two curves.

The required area is equal to

$$2\,(m^{\frac{1}{2}} \sim \mu^{\frac{1}{2}}).a^{\frac{3}{2}}.$$

4. To find the magnitude of the ordinate of such a point in a parabola, that the intercepts, on the axes of coordinates, of a tangent drawn to the curve at this point, may be equal to each other.

The required magnitude = half the latus-rectum.

SECTION III.

Referred to the Axis and its Tangent. Magical Equation to the Tangent.

1. A straight line, inclined at an angle ϕ to the axis of x, touches both the curves

$$y^2 = 4mx, \quad x^2 + y^2 = c^2:$$

to find the value of ϕ.

Put $\tan\phi = \alpha$: then the equation to the line, since it touches the former curve, will be

$$y = \alpha x + \frac{m}{\alpha},$$

and, since it touches the latter,

$$y = \alpha x + c(1 + \alpha^2)^{\frac{1}{2}}.$$

Since these two equations represent the same line, they must be identical; hence

$$c(1 + \alpha^2)^{\frac{1}{2}} = \frac{m}{\alpha},$$

whence $$c \sec\phi = m \cot\phi,$$

or $$c \sin\phi = m \cos^2\phi;$$

an equation which determines the value of ϕ.

2. Two straight lines, which are always tangents to a given parabola, are such that the sum of the cotangents of their inclinations to its axis is constant; to find the equation to the locus of their intersection.

The equation to either tangent is

$$y = ax + \frac{m}{a},$$

whence
$$\frac{1}{a^2} - \frac{y}{m}\frac{1}{a} + \frac{x}{m} = 0.$$

Hence, a', a'', denoting the two values of a,

$$\frac{1}{a'} + \frac{1}{a''} = \frac{y}{m}:$$

but, by the condition of the problem,

$$\frac{1}{a'} + \frac{1}{a''} = \beta,$$

β denoting some constant quantity.

Hence
$$y = m\beta,$$

the equation to the required locus, which is therefore a diameter of the parabola.

3. If α_1, α_2, be the trigonometrical tangents of the inclinations of any two tangents of a parabola to its axis, to find the equation to the tangent at the extremity of the diameter which passes through their intersection.

The equation to the parabola being

$$y^2 = 4mx,$$

the equations to the first two tangents will be

$$y = \alpha_1 x + \frac{m}{\alpha_1},$$

and
$$y = \alpha_2 x + \frac{m}{\alpha_2}.$$

At their point of intersection, y_1 being its ordinate,

$$y_1 (\alpha_2 - \alpha_1) = m\left(\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{\alpha_2}\right);$$

and therefore
$$y_1 = \frac{m}{\alpha_1 \alpha_2}(\alpha_1 + \alpha_2),$$

which is the equation to the diameter

The equation to the required tangent will therefore be

$$y = \frac{2m}{y_1}(x + x_1)$$

$$= \frac{2m}{y_1}x + \tfrac{1}{2}y_1,$$

or
$$y = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}x + \frac{\alpha_1 + \alpha_2}{2\alpha_1\alpha_2}m.$$

4. From the vertex of a parabola a straight line is drawn, inclined at an angle $\frac{1}{4}\pi$ to the tangent at any point; to find the equation to the locus of their intersection.

Let the equation to the straight line be

$$y = \alpha' x \text{..........................} (1);$$

then, by the hypothesis, the equation to the tangent being

$$y = \alpha x + \frac{m}{\alpha} \text{..........................} (2),$$

we shall have
$$1 = \frac{\alpha' - \alpha}{1 + \alpha'\alpha}, \quad \alpha' = \frac{1 + \alpha}{1 - \alpha};$$

hence the equation (1) becomes

$$y = \frac{1 + \alpha}{1 - \alpha}x \text{..........................} (3).$$

Obtaining α from (3), in terms of x and y, and substituting its value in (2), we shall get

$$y = x\frac{y - x}{y + x} + m\frac{y + x}{y - x},$$

or
$$(y^2 + x^2).(y - x) = m(y + x)^2,$$

which is the equation to the required locus.

5. To prove that the three altitudes of any triangle circumscribed about a parabola all pass through a single point in the directrix.

The equations to the three sides of the triangle will be of the forms

$$y = \alpha x + \frac{m}{\alpha} \text{......} (1), \quad y = \alpha' x + \frac{m}{\alpha'} \text{......} (2), \quad y = \alpha'' x + \frac{m}{\alpha''} \text{......} (3).$$

The equation to one altitude, viz. that which passes through the intersection of (1) and (2) at right angles to (3), will be, as may easily be ascertained,

$$\alpha\alpha'(x + \alpha''y) = m(1 + \alpha'\alpha'' + \alpha''\alpha) \dots\dots\dots\dots (4).$$

By similarity it is clear that the equation to the altitude, which is at right angles to (1), is

$$\alpha'\alpha''(x + \alpha y) = m(1 + \alpha''\alpha + \alpha\alpha') \dots\dots\dots\dots (5).$$

At the intersection of (4) and (5)

$$x = -m, \qquad y = \frac{m}{\alpha\alpha'\alpha''}(1 + \alpha'\alpha'' + \alpha''\alpha + \alpha\alpha').$$

These results, being symmetrical in relation to α, α', α'', shew that the three altitudes all pass through a single point. The value of x shews that the point lies in the directrix, which is therefore the locus of the intersection of the altitudes of a variable tangential triangle.

Steiner: *Gergonne*, *Annales de Mathématiques*, tom. XIX. p. 59.

6. To find the equation to the tangent of a parabola $y^2 = 4mx$, which passes through the point in which the directrix cuts the axis.

The required equation is

$$y = x + m.$$

7. To find the locus of the intersection of two straight lines, which always touch a parabola

$$y^2 = 4mx;$$

the product of the trigonometrical tangents of their inclinations to the axis of x being a constant quantity k.

The locus is a straight line of which the equation is

$$x = \frac{m}{k^2}.$$

Rochat: *Gergonne*, *Annales de Mathématiques*, tom. II. p. 229.

8. To find the distances of the vertex and focus of a parabola from the tangent, in terms of the inclination of the tangent to the axis of x.

If ϕ denote the inclination, the required distances are respectively equal to

$$\frac{m\cos^2\phi}{\sin\phi} \text{ and } \frac{m}{\sin\phi}.$$

9. If, from the focus of a parabola, lines be drawn to meet the tangents at a constant angle, to prove that the locus of the points of intersection will be that tangent to the parabola, the inclination of which to the axis is equal to the given angle.

10. Two tangents to a parabola make angles θ, θ', with its axis. To find the locus of their intersection, having given that $\sin\theta.\sin\theta'$ is invariable in magnitude.

Put $\sin\theta.\sin\theta' = \alpha$; then, $y^2 = 4mx$ being the equation to the parabola, the required locus will be a circle defined by the equation

$$(x-m)^2 + y^2 = \frac{m^2}{\alpha^2}.$$

11. Two tangents to a parabola make angles θ, θ', with its axis. To find the locus of their intersection, supposing $\cot\theta - \cot\theta'$ to be invariable.

The equation to the parabola being $y^2 = 4mx$, the required locus will be another parabola, the equation to which, α denoting $\cot\theta - \cot\theta'$, is

$$y^2 = 4mx + \alpha^2 m^2.$$

12. To prove that the area of the triangle formed by three tangents to a parabola, the cotangents of the inclinations of which to the axis are successive terms of an arithmetical progression, of which the common difference is λ, is equal to $\lambda^3 m^2$, $4m$ being the latus-rectum.

13. From two points in the diameter of a parabola, two pairs of tangents are drawn to the curve; the trigonometrical tangents of the inclinations of the one pair to the axis are α_1, α_2, and, of the other pair, α_3, α_4: to prove that

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{1}{\alpha_3} + \frac{1}{\alpha_4}.$$

14. If any hexagon be described about a parabola, to prove that its three diagonals will all pass through a single point. (This problem is a particular case of the same proposition in regard to any conic section).

Lubbock: *Philosophical Magazine*, *August* 1838.
Ellis: *Cambridge Mathematical Journal*, vol. I. p. 204.

15. If x_1, y_1; x_2, y_2; x_3, y_3; ...... be the coordinates of the angles of any re-entering polygon of $2n$ sides, circumscribing a parabola, to prove that

$$x_1 x_3 x_5 \ldots\ldots x_{2n-1} = x_2 x_4 x_6 \ldots\ldots x_{2n},$$

and

$$y_1 - y_2 + y_3 - \ldots\ldots - y_{2n} = 0.$$

Ellis: *Cambridge Mathematical Journal*, vol. II. p. 48.

16. To prove that the continued product of the abscissæ of the points of intersection of any number of tangents to a parabola, is equal to the continued product of the abscissæ of the points of contact, provided that no three points of intersection lie in the same straight line.

Ellis: *Cambridge Mathematical Journal*, vol. II. p. 48.

SECTION IV.

Referred to the Axis and its Tangent. Normals.

1. In a parabola $y^2 = lx$, the ordinates of three points, such that the normals pass through the same point, are y_1, y_2, y_3: to prove that

$$y_1 + y_2 + y_3 = 0,$$

and to find the equation to the circle passing through these three points.

Let (x, y) be the coordinates of any one of the three points; α, β, being the common point of the normals. Then

$$\beta - y = -\frac{2y}{l}(\alpha - x).$$

Eliminating x between this equation and $y^2 = lx$, we have

$$2y^3 = l(2\alpha - l)\,y + l^2\beta \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The form of this cubic shews that

$$y_1 + y_2 + y_3 = 0.$$

Let the equation to the circle passing through the three points be

$$x^2 + y^2 + \lambda x + \mu y + \nu = 0.$$

From this equation and that to the parabola, we have, for the points of intersection,

$$y^4 + l(l + \lambda)\, y^2 + l^2 \mu y + l^2 \nu = 0 \text{ ....(2)}.$$

Multiplying this equation by 2 and attending to (1), we get

$$l(2\alpha + l + 2\lambda)\, y^2 + l^2(\beta + 2\mu)\, y + 2l^2\nu = 0.$$

This equation must be satisfied by y_1, y_2, y_3, as roots: hence we must have

$$2\alpha + l + 2\lambda = 0, \quad \beta + 2\mu = 0, \quad \nu = 0.$$

These three equations reduce the equation to the circle to the form

$$x^2 + y^2 - (\alpha + \tfrac{1}{2}l)\, x - \tfrac{1}{2}\beta y = 0.$$

This equation shews that the circle passes through the vertex of the parabola.

2. To find the locus of the intersections of the normals at any two points of a parabola, on opposite sides of its axis, the ordinates of which are as 1 to 2.

The equations to such a pair of normals are

$$y - y' = -\frac{y'}{2m}\left(x - \frac{y'^2}{4m}\right) \text{..................(1)},$$

$$y + 2y' = \frac{y'}{m}\left(x - \frac{y'^2}{m}\right) \text{ .... (2)}.$$

Multiplying the equation (1) by 8 and adding the result to the equation (2), we have, at the intersection of these two lines,

$$9y - 6y' = -\frac{3y'}{m}\, x,$$

$$y' = \frac{3my}{2m - x}.$$

Substituting this expression for y' in (1), and reducing, we shall get for the equation to the required locus

$$y^2 = \frac{4}{27m}(x - 2m)^3,$$

which is the equation to the evolute of the parabola.

3. If SL be drawn from the focus S of a parabola perpendicular to the normal at any point P, to find the locus of the point L.

If $\tan^{-1}\alpha$ be the angle made by the tangent at P with the axis, the coordinates of P will be

$$\frac{m}{\alpha^2}, \quad \frac{2m}{\alpha}.$$

Hence the equation to the normal at P will be

$$y - \frac{2m}{\alpha} = -\frac{1}{\alpha}\left(x - \frac{m}{\alpha^2}\right) \dots\dots\dots\dots\dots\dots(1).$$

Also, the equation to SL will be

$$y = \alpha(x - m) \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

From (1) we have

$$\alpha^2 y + \alpha x = m\left(\frac{1}{\alpha} + 2\alpha\right),$$

and, from (2), $\quad \alpha x - y = \alpha m.$

Hence, at L, $\quad (1 + \alpha^2)\, y = m\left(\frac{1}{\alpha} + \alpha\right),$

$$y = \frac{m}{\alpha};$$

and therefore, from (2),

$$y^2 = m(x - m),$$

which is the equation to the locus of L. The locus is therefore a parabola the latus-rectum of which is one-fourth of that of the original parabola and of which the vertex is S.

4. To find the equation to the normal of a parabola, which is inclined at any proposed angle to the axis of the curve.

If α = the tangent of the normal's inclination to the axis of the parabola, the equation required will be

$$\alpha x = y + (\alpha^2 + 2)\,\alpha m.$$

5. To prove that, $y^2 = 4mx$ being the equation to a parabola, three normals may be drawn to the curve from any point within the area of the curve defined by the equation

$$27my^2 = 4\,(x - 2m)^3,$$

and, from any point without the area, only one.

6. Two normals to a parabola are always at right angles to each other: to find the locus of their intersection.

The equation to the parabola being $y^2 = 4mx$, the required locus will be another parabola defined by the equation

$$y^2 = m\,(x - 3m).$$

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVII. p. 281.

7. To draw, from a given point, normals to a parabola.

Let (a, b) be the coordinates of the given point, and let $y^2 = 4mx$ be the equation to the parabola. There will be generally three normals passing through the intersections of the parabola with the circle represented by the equation

$$x^2 - (2m + a)\,x + y^2 - \tfrac{1}{2}by = 0.$$

James Bernoulli: *Analysis et Constructio Problematis Hugeniani*, *Opera*, tom. II. p. 700.

8. The axis Ax of a parabola is divided into a number of successive portions AM_1, M_1M_2, M_2M_3,...... equal respectively to l, $3l$, $5l$, ... where l denotes the latus-rectum. Circles are described with AM_1, M_1M_2, M_2M_3,...... as diameters. To prove that two radii of every circle, except the first, are normals to the parabola.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 43.

SECTION V.

Referred to the Axis and its Tangent. Chords.

1. If chords be drawn to a parabola, all passing through the point where the axis meets the directrix, to find the equation to the locus of their middle points.

Let (x_1, y_1), (x_2, y_2), be the two extremities of any one of the chords. Then, y denoting the ordinate of its middle point,

$$y = \tfrac{1}{2}(y_1 + y_2) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now y_1, y_2, are the roots of a quadratic resulting from the elimination of x between the equations

$$y = \alpha(x + m) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and

$$y^2 = 4mx;$$

that is, of the equation

$$y^2 - \frac{4m}{\alpha} y + 4m^2 = 0:$$

hence

$$y_1 + y_2 = \frac{4m}{\alpha},$$

and therefore, by (1),

$$y = \frac{2m}{\alpha} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating α between (2) and (3), we get for the equation to the locus,

$$y^2 = 2m(x + m),$$

which belongs to a parabola similar to the original one; the distances between their vertices being m.

2. If a circle, described upon a chord of a parabola as a diameter, meets the directrix, to prove that it also touches it; and to shew that all the chords, for which this is possible, intersect in a single point.

Let (x', y'), (x'', y''), be the two ends of the diameter. Then the equation to the circle will be

$$\{x - \tfrac{1}{2}(x' + x'')\}^2 + \{y - \tfrac{1}{2}(y' + y'')\}^2 = \tfrac{1}{4}\{(x'' - x')^2 + (y'' - y')^2\},$$

or

$$x^2 + y^2 - (x' + x'')\,x - (y' + y'')\,y + x'x'' + y'y'' = 0.$$

Let the equations to the chord and parabola be

$$y = \alpha x + \beta, \qquad y^2 = 4mx.$$

At their intersection,

$$\alpha^2 x^2 - 2(2m - \alpha\beta)\, x + \beta^2 = 0,$$

and $$\alpha y^2 - 4my + 4m\beta = 0:$$

hence $$x' + x'' = \frac{2}{\alpha^2}(2m - \alpha\beta), \qquad x'x'' = \frac{\beta^2}{\alpha^2},$$

$$y' + y'' = \frac{4m}{\alpha}, \qquad y'y'' = \frac{4m\beta}{\alpha}.$$

The equation to the circle is therefore

$$x^2 + y^2 + \frac{2}{\alpha^2}(\alpha\beta - 2m)\, x - \frac{4m}{\alpha}\, y + \frac{\beta^2}{\alpha^2} + \frac{4m\beta}{\alpha} = 0.$$

If the circle meets the directrix, $x = -m$, and therefore

$$y^2 - \frac{4m}{\alpha}\, y + \frac{4m^2}{\alpha^2} + \left(m + \frac{\beta}{\alpha}\right)^2 = 0 \text{............}(1).$$

In order that y may be possible, we must have

$$m + \frac{\beta}{\alpha} = 0 \text{ ....}(2),$$

which shews that the equation to the chord is

$$y = \alpha(x - m);$$

the chords therefore all pass through the focus.

The roots of (1), under the condition (2), are equal, and therefore the circle *touches* when it *meets* the directrix.

3. To find the equations to all the common chords of the two curves $$y^2 = 2cx - x^2, \qquad y^2 = 4mx.$$

The required equations are

$$x = 2(c - 2m), \qquad y = \pm\left(\frac{2m}{c - 2m}\right)^{\frac{1}{2}} . \, x.$$

4. To inscribe in a given parabola a chord Pp of given length, passing through a given point Q.

Let Pp intersect the axis Ax of the parabola in O, and let QN be the ordinate of Q. Let $AN = a$, $QN = b$, $Pp = c$, $ON = z$, l = the latus-rectum. Then the value of z will be defined by the biquadratic equation

$$l^2z^4 - 4b^2lz^3 + b^2l(4a+l)z^2 + b^4(4al - c^2) = 0.$$

Newton: *Arithmetica Universalis*, prob. XIV.

SECTION VI.

Referred to the Axis and its Tangent. Focal Properties.

1. P', P'', being any two points in a parabola, and O the point of concourse of the tangents at these points, and S the focus: to prove that

$$\frac{(P'O)^2}{P'S} = \frac{(P''O)^2}{P''S}.$$

Let the equations to the tangents at P', P'', be

$$y = \alpha' x + \frac{m}{\alpha'},$$

$$y = \alpha'' x + \frac{m}{\alpha''}.$$

Combining these two equations we obtain, for the coordinates of the point O,

$$x_1 = \frac{m}{\alpha'\alpha''}, \qquad y_1 = \frac{m}{\alpha'\alpha''}\cdot(\alpha' + \alpha'').$$

Also, combining each of the equations to the tangents with that to the parabola, viz. $y^2 = 4mx$, we have, for the coordinates of P', P'', respectively,

$$\left\{\begin{array}{l} x' = \dfrac{m}{\alpha'^2} \\[2ex] y' = \dfrac{2m}{\alpha'} \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l} x'' = \dfrac{m}{\alpha''^2} \\[2ex] y'' = \dfrac{2m}{\alpha''} \end{array}\right\}.$$

Hence $$(P'O)^2 = (x_1 - x')^2 + (y_1 - y')^2,$$

$$= m^2\left(1 + \frac{1}{\alpha'^2}\right)\left(\frac{1}{\alpha'} - \frac{1}{\alpha''}\right)^2,$$

and
$$(P'S)^2 = (x' - m)^2 + y'^2$$
$$= m^2\left(1 + \frac{1}{\alpha'^2}\right)^2.$$

We have therefore
$$\frac{(P'O)^2}{P'S} = m\left(\frac{1}{\alpha'} - \frac{1}{\alpha''}\right)^2.$$

Symmetry shews that we must have also
$$\frac{(P''O)^2}{P''S} = m\left(\frac{1}{\alpha''} - \frac{1}{\alpha'}\right)^2.$$

Hence
$$\frac{(P'O)^2}{P'S} = \frac{(P''O)^2}{P''S}.$$

Massabieau, Guillaume, Gobert, Bérard: *Gergonne, Annales de Mathématiques*, tom. IV. p. 183.

2. To find the locus of the intersection of a tangent at one extremity of a focal chord of a parabola with the ordinate at the other produced.

Let x', y', be the coordinates of the point of contact of the tangent; and let (x, y) be the point of intersection of the tangent and ordinate. Then
$$yy' = 2m(x + x'),$$
$$y^2y'^2 = 4m^2(x + x')^2,$$
$$x'y^2 = m(x + x')^2 \quad\text{......................}(1).$$

But
$$\frac{1}{m} = \frac{1}{SP} + \frac{1}{SP'} = \frac{1}{m + x'} + \frac{1}{m + x},$$

whence
$$\frac{1}{m + x'} = \frac{x}{m(m + x)},$$

and
$$x' = \frac{m^2}{x}.$$

Hence, from (1),
$$\frac{m^2y^2}{x} = m\left(x + \frac{m^2}{x}\right)^2,$$
$$mxy^2 = (x^2 + m^2)^2,$$

the equation to the required locus.

3. To prove that a series of circles, of which the centres are in a parabola, and which pass through the focus, all touch the directrix.

4. PT, PG, are the tangent and normal of a parabola, at a point P, terminating at T, G, in the axis: SQ, SR, are perpendiculars upon PT, PG, respectively, from the focus S: to find the area of the rectangle $SQPR$, and to determine the position of P when the rectangle is a square.

The equation to the parabola being $y^2 = 4mx$, the area of $SQPR$, if x, y, are the coordinates of P, is equal to

$$\tfrac{1}{2}y\,(x+m)\,;$$

and, when the rectangle is a square, the point P is at an extremity of the latus-rectum.

5. If from the focus of a parabola as centre and with the focal distance of a point in the parabola as radius, a circle be described, to prove that the intersections of the tangent and normal to the parabola at the point, with the axis, will lie in the circumference of the circle.

6. If SL be drawn, from the focus S of a parabola, at right angles to the normal at any point P: to prove that the abscissa of the point L is equal to SP, and that, AM being the abscissa of P,

$$(SL)^2 = AM.SP.$$

7. In the preceding problem, to find the locus of the point L.

The equation to the parabola being $y^2 = 4mx$, the required locus is another parabola defined by the equation

$$y^2 = m\,(x-m).$$

SECTION VII.

Referred to a Tangent and its diameter as Axes.

1. From a point O are drawn two lines to touch a parabola in the points P and Q: another line, parallel to PQ, touches the parabola in R and intersects OP, OQ, in S, T, respectively: if V be the intersection of the lines joining PT, QS, crosswise, to prove that O, R, V, are in the same straight line.

Let RT be taken as the axis of y, and the diameter through R as the axis of x. Since PQ is parallel to SRT, we may take (x', y') and $(x', -y')$, as the coordinates of Q, P, respectively.

The equations to QO, PO, will accordingly be of the forms

$$y = \frac{2m}{y'}(x + x')$$

and

$$y = -\frac{2m}{y'}(x + x').$$

These two lines therefore intersect in the axis of x.

Moreover the coordinates of T are

$$x_1 = 0,$$

$$y_1 = \frac{2mx'}{y'} = \tfrac{1}{2}y'.$$

The equation to PT is therefore

$$x'y + \tfrac{3}{2}y'x = \tfrac{1}{2}x'y'.$$

Similarly, putting $-y'$ for y', we see that the equation to QS is

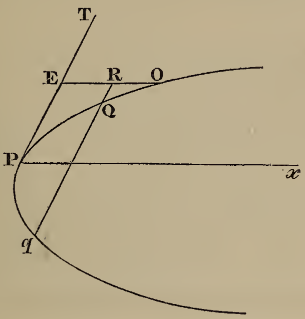
$$x'y - \tfrac{3}{2}y'x = -\tfrac{1}{2}x'y'.$$

These two lines PT, QS, therefore also intersect in the axis of x. Thus O, R, V, all lie in the axis of x.

2. From any point O, in the arc of a parabola, a straight line OR is drawn, parallel to the axis of the curve, to meet a chord Qq, produced if necessary, in R. To prove that

$$RQ.Rq \propto RO.$$

Let a tangent PT, and its diameter Px, be taken as axes of coordinates, PT being parallel to qQ.



Produce OR to meet PT in E.

The equation to the curve is of the form

$$y^2 = lx.$$

Let h, k, be the coordinates of R: then, if we put h for x, and $k - y$ for y, the values of y in the resulting quadratic

$$(y-k)^2 = lh,$$

or
$$y^2 - 2ky + k^2 - lh = 0,$$

will be equal to RQ, Rq: hence

$$\begin{aligned} RQ.Rq &= k^2 - lh, \\ &= l(OE - RE), \\ &= l.RO, \\ &\propto RO. \end{aligned}$$

De la Hire: *Sectiones Conicæ*, lib. V. prop. I.

3. If two tangents be drawn to a parabola, to prove that a third tangent, parallel to the chord joining the points of contact, will bisect the parts of the other tangents which are included between their point of intersection and their points of contact.

4. BC, CD, are two consecutive arcs of a parabola, the sagittæ of which, bisecting their chords and being parallel to the axis, are equal; to prove that the chord BD is parallel to the tangent at C.

5. From a point B, without a parabola, two straight lines are drawn, one touching the curve in P, and the other cutting it in Q. A straight line HEF is drawn, parallel to BP, cutting in the point H the diameter through B, and the curve in the points E, F. HQ is joined. To prove that

$$(BP)^2 : HE.EF :: BQ : HQ.$$

De la Hire: *Sectiones Conicæ*, lib. III. prop. 28.

6. Qq is a chord of a parabola: from any point R in this chord is drawn REF, parallel to the axis of the parabola, meeting the curve in E, and the tangent, drawn through the point Q, in F: to prove that

$$RQ : Rq :: EF : ER.$$

De la Hire: *Sectiones Conicæ*, lib. V. prop. 25.

7. Through a point P in a parabola, a chord is drawn to cut in Q, q, another parabola, equal to the former, the axes of the two curves lying in a single straight line: to prove that, Qq remaining always parallel to itself, the rectangle $PQ.Pq$ will remain constant for all positions of P.

De la Hire: *Sectiones Conicæ*, lib. VI. prop. 13.

SECTION VIII.

Referred to two Tangents as Axes.

If the parabola be referred to two tangents as axes of co-ordinates, its equation will assume the symmetrical form

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} \pm \left(\frac{y}{b}\right)^{\frac{1}{2}} = \pm 1,$$

a and b being the lengths of the tangents from their intersection to the points of contact.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 105, *troisième édition.*

Gregory: *Cambridge Mathematical Journal*, vol. II. p. 14.

1. To prove that two parabolas cannot touch each other in more than one point.

If possible, let the tangents at two points of contact be taken as axes of coordinates, and let a, b, be the distances of these points of contact from the origin.

Then the equation to either parabola will be

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} \pm \left(\frac{y}{b}\right)^{\frac{1}{2}} = \pm 1;$$

and therefore the two parabolas coincide.

Παραβολὴ παραβολῆς οὐκ ἐφάπτεται κατὰ πλείονα σημεῖα ἢ ἕν.

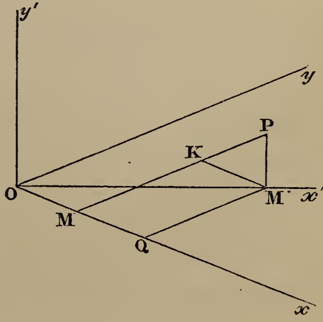
'ΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ *Κωνικῶν τὸ τέταρτον, Πρότασις κή.*

2. The equation to a curve referred to oblique axes, which are inclined to one another at an angle θ, is

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}:$$

to prove that, when the axes are changed into a system of rectangular ones, of which the axis of x bisects θ, the equation will become

$$y^2 = \frac{2a \sin^2 \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} \left(x - \tfrac{1}{2} a \cos \tfrac{1}{2}\theta\right).$$



Let P be any point whatever. Let OM, PM, be the coordinates of P in regard to the original axes Ox, Oy; and OM', PM', in regard to the new axes Ox', Oy'. Draw $M'K$, parallel to xO, meeting PM in K.

Then $$x = OQ - M'K$$

$$= x' \frac{\sin \frac{1}{2}\theta}{\sin \theta} - y' \frac{\cos \frac{1}{2}\theta}{\sin \theta}$$

$$= \frac{x'}{2 \cos \frac{1}{2}\theta} - \frac{y'}{2 \sin \frac{1}{2}\theta}.$$

Also $$y = M'Q + KP = x' \frac{\sin \frac{1}{2}\theta}{\sin \theta} + y' \frac{\cos \frac{1}{2}\theta}{\sin \theta}$$

$$= \frac{x'}{2 \cos \frac{1}{2}\theta} + \frac{y'}{2 \sin \frac{1}{2}\theta}.$$

Now $$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}},$$

and therefore $$x + y + 2 (xy)^{\frac{1}{2}} = a:$$

hence $$\frac{x'}{\cos \frac{1}{2}\theta} + \left(\frac{x'^2}{\cos^2 \frac{1}{2}\theta} - \frac{y'^2}{\sin^2 \frac{1}{2}\theta}\right)^{\frac{1}{2}} = a,$$

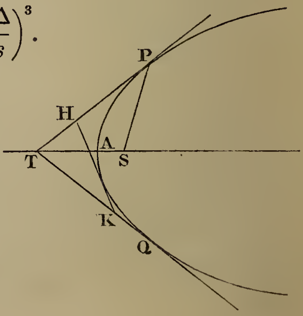
$$\frac{x'^2}{\cos^2 \frac{1}{2}\theta} - \frac{y'^2}{\sin^2 \frac{1}{2}\theta} = \left(a - \frac{x'}{\cos \frac{1}{2}\theta}\right)^2,$$

$$y'^2 = \frac{2a \sin^2 \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} (x' - \tfrac{1}{2} a \cos \tfrac{1}{2}\theta).$$

3. If three parabolas be described, having their principal diameters in the continuations of the lines bisecting the three angles of any plane triangle, and if each parabola touch one side of the triangle and the prolongation of the other two: then, if Δ denote the area of the triangle; a, b, c, its sides; s half the sum of the sides; and π_1, π_2, π_3, the semiparameters of the three parabolas, it is required to prove that

$$\pi_1 \pi_2 \pi_3 = \frac{(b + c)(c + a)(a + b)}{abc} \cdot \left(\frac{\Delta}{s}\right)^3.$$

Let HKT be the triangle. If S be the focus of the parabola in the diagram, TS will bisect the angle $A = PTK$. If P, Q, be the points of contact, it is plain that $PT = QT$. Let TP, TQ, be taken as axes of coordinates. Then the equation to



the parabola, if $TP = m$, will be

$$x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm m^{\frac{1}{2}} \dots\dots\dots\dots (1).$$

Let $TH = a$, $TK = b$. Then

$$\frac{x}{a} + \frac{y}{b} = 1 \dots\dots\dots\dots (2)$$

is the equation to a tangent to the parabola.

But the tangent at a point x_1, y_1, is

$$\frac{x}{(mx_1)^{\frac{1}{2}}} \pm \frac{y}{(my_1)^{\frac{1}{2}}} = \pm 1 \dots\dots\dots\dots (3).$$

Comparing (2) and (3), on the supposition of their representing the same line, we see that

$$(mx_1)^{\frac{1}{2}} = \pm a, \qquad (my_1)^{\frac{1}{2}} = b:$$

and therefore, by virtue of (1), putting x_1, y_1, for x, y, we get

$$a + b = m = PT.$$

But, by known properties of the parabola,

$$PT = 2SP \cos \tfrac{1}{2}A,$$

and $$PT \sin^2 \tfrac{1}{2}A = 2AS \cos \tfrac{1}{2}A;$$

hence $$\pi_1 = 2AS = (a + b) \frac{\sin^2 \frac{1}{2}A}{\cos \frac{1}{2}A}.$$

Similarly $$\pi_2 = (b + c) \frac{\sin^2 \frac{1}{2}B}{\cos \frac{1}{2}B},$$

and $$\pi_3 = (c + a) \cdot \frac{\sin^2 \frac{1}{2}C}{\cos \frac{1}{2}C}.$$

Hence $$\pi_1 \pi_2 \pi_3 = \frac{(b+c)(c+a)(a+b)}{abc} \cdot \frac{\{s(s-a)(s-b)(s-c)\}^{\frac{3}{2}}}{s^3}$$

$$= \frac{(b+c)(c+a)(a+b)}{abc} \cdot \left(\frac{\Delta}{s}\right)^3.$$

Lady's and Gentleman's Diary, 1844.

4. From P, the point of concourse of two tangents to a parabola, viz. PQ, PQ', a straight line $PABC$ is drawn, meeting

the curve in A, C, and the chord QQ' in B. To shew that PA, PB, PC, are in harmonical progression.

Let $PQ = a$, $PQ' = b$. Then the equation to the parabola, referred to these tangents as axes, will be

$$\left(\frac{y}{b}\right)^{\frac{1}{2}} \pm \left(\frac{x}{a}\right)^{\frac{1}{2}} = \pm 1 \text{ .... } (1).$$

The equation to QQ' will be

$$\frac{x}{a} + \frac{y}{b} = 1 \text{.........................} (2).$$

The equation to $PABC$ we may take to be

$$y = \beta x \text{ .... } (3).$$

Let x_1, x_2, x_3, be the abscissæ of A, B, C, respectively: then, from (1) and (3),

$$\left(\frac{\beta x}{b}\right)^{\frac{1}{2}} \pm \left(\frac{x}{a}\right)^{\frac{1}{2}} = \pm 1, \quad \frac{1}{x} = \left\{\left(\frac{1}{a}\right)^{\frac{1}{2}} \pm \left(\frac{\beta}{b}\right)^{\frac{1}{2}}\right\};$$

whence $$\frac{1}{x_1} = \left\{\left(\frac{1}{a}\right)^{\frac{1}{2}} + \left(\frac{\beta}{b}\right)^{\frac{1}{2}}\right\}^2, \quad \frac{1}{x_3} = \left\{\left(\frac{1}{a}\right)^{\frac{1}{2}} - \left(\frac{\beta}{b}\right)^{\frac{1}{2}}\right\}^2;$$

and, from (2) and (3), $$\frac{1}{x_2} = \frac{1}{a} + \frac{\beta}{b}.$$

Hence we see that $$\frac{2}{x_2} = \frac{1}{x_1} + \frac{1}{x_3}.$$

But it is plain that x_1, x_2, x_3, are proportional to PA, PB, PC; hence

$$\frac{2}{PB} = \frac{1}{PA} + \frac{1}{PC}.$$

De la Hire: *Sectiones Conicæ*, lib. II. prop. 21.

5. To determine the magnitude of the latus-rectum of the parabola represented by the equation

$$x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm a^{\frac{1}{2}},$$

the axes being rectangular.

The latus-rectum $= a\sqrt{2}$.

6. TP, TQ, are two tangents to a parabola: any other tangent cuts these two in points p, q, respectively: to prove that

$$\frac{Tp}{TP} + \frac{Tq}{TQ} = 1.$$

7. PT, QT, are two equal tangents to a parabola, P and Q being the points of contact. If PT, QT, be cut by a third tangent in E, F, respectively, to prove that

$$PE = FT, \text{ and } QF = ET.$$

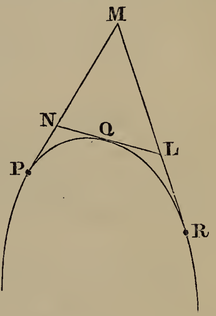
8. If there are three tangents to a parabola, the triangle formed by their intersections is half of that the angular points of which are the points of contact.

Gregory: *Cambridge Mathematical Journal*, vol. II. p. 16.

9. Three straight lines MN, NL, LM, produced if necessary, touch a parabola in the points P, Q, R, respectively; to prove that

$$RL : LM :: LQ : QN :: MN : NP.$$

De la Hire: *Sectiones Conicæ*, lib. III. prop. 20.



10. Two tangents OA, OB, are drawn to a parabola: in the line AB, joining the points of contact A, B, is taken any point whatever E: from E are drawn the straight lines EH, EK, parallel to BO, AO, respectively, and cutting AO, BO, respectively in H, K: to prove that HK will touch the parabola in some point P, and that PE is a diameter of the curve.

De la Hire: *Sectiones Conicæ*, lib. III. prop. 21.

11. Two straight lines OA, OB, touch a parabola in A, B. The chord AB is joined. M is the middle point of AB; OM is joined. A third tangent to the parabola cuts OA, OB, respectively in H, K.

If $\angle MOA = \lambda$, $\angle MOB = \mu$, $OA = a$, $OB = b$, $OH = \alpha$, $OK = \beta$, to prove that the equation to the directrix, OA, OB, being chosen as the axes of x, y, respectively, will be

$$x \cos\lambda + y \cos\mu = \cot(\lambda + \mu).(\alpha \sin\lambda + \beta \sin\mu).$$

Cor. It may easily be shewn that the directrix always passes through a fixed point of which, ω denoting the angle between the axes, the coordinates are equal to

$$\frac{\cos\omega}{\sin^2\omega}(\beta - \alpha\cos\omega), \quad \frac{\cos\omega}{\sin^2\omega}(\alpha - \beta\cos\omega).$$

12. Two tangents to a parabola, the lengths of which are a, b, intersect in O at an angle ω, and a circle is inscribed between the tangents and the curve; to find the distance of the centre of the circle from O.

The required distance $= \dfrac{ab}{(a+b)\sec\frac{1}{2}\omega + 2(ab)^{\frac{1}{2}}.\tan\frac{1}{2}\omega}$.

13. Parabolas are described touching two given straight lines at right angles to each other; to find the locus of the vertices of the parabolas, supposing the chords of contact to be all parallel to a fixed line.

Let the two given straight lines be taken as axes of coordinates; then, l, m, being the direction-cosines of the fixed line, the locus of the vertices will be a straight line represented by the equation

$$\frac{x}{l^3} = \frac{y}{m^3}.$$

Section IX.

Referred to any Rectangular Axes whatever. Reduction.

The most general form of the equation to a parabola is

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

the parameters a, b, c, being subject to the relation $c^2 = ab$.

The object of the general problem of reduction is to ascertain the positions of the vertex of the parabola and its focus. This object may be effected by first turning the axes of coordinates through such an angle that the coefficient of xy in the transformed equation may be zero, and then changing the origin into such a position that the final equation may consist of only two terms, one term containing the first power of one of the coordinates, and the other term containing the second power of the other. The equation will then be reduced to the more simple form belonging to the axis and its tangent as axes of coordinates.

1. To find the position of the vertex and the magnitude of the latus-rectum of the parabola

$$x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm a^{\frac{1}{2}}.$$

Squaring the equation, we have

$$x \pm 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = a,$$

and thence
$$(x + y - a)^2 = 4xy,$$

or
$$x^2 - 2xy + y^2 - 2ax - 2ay + a^2 = 0.$$

If we turn the axes through an angle θ, the equation will become

$$(x' \cos\theta - y' \sin\theta)^2 - 2(x' \cos\theta - y' \sin\theta).(x' \sin\theta + y' \cos\theta)$$
$$+ (x' \sin\theta + y' \cos\theta)^2 - 2a(x' \cos\theta - y' \sin\theta)$$
$$- 2a(x' \sin\theta + y' \cos\theta) + a^2 = 0.$$

Assume the coefficient of $x'y'$ to be equal to zero; then

$$-2 \sin\theta.\cos\theta - 2(\cos^2\theta - \sin^2\theta) + 2 \sin\theta.\cos\theta = 0,$$

and therefore $\cos 2\theta = 0, \quad 2\theta = \frac{1}{2}\pi, \quad \theta = \frac{1}{4}\pi.$

Hence, substituting for $\cos\theta$ and $\sin\theta$ their common value $\frac{1}{\sqrt{2}}$, in the equation to the curve, we get

$$\tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 - 2(\tfrac{1}{2}x'^2 - \tfrac{1}{2}y'^2) + \tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2$$
$$- a(x'\sqrt{2} - y'\sqrt{2}) - a(x'\sqrt{2} + y'\sqrt{2}) + a^2 = 0,$$

and therefore
$$2y'^2 - 2\sqrt{2}\,ax' + a^2 = 0.$$

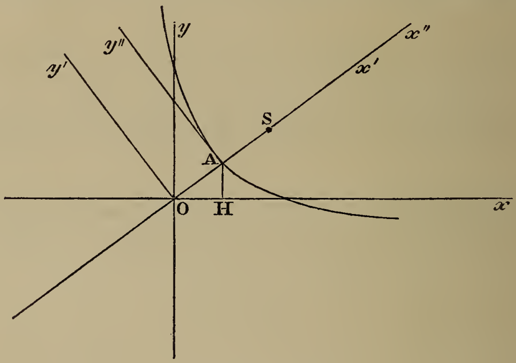
Again, changing the origin of coordinates to a point α, β, we have
$$2(y''+\beta)^2 - 2\sqrt{2}.a(x''+\alpha) + a^2 = 0.$$

Equating to zero the coefficient of y'' and those terms which involve neither x'' nor y'', we obtain
$$4\beta = 0, \quad \beta = 0,$$
and
$$2\beta^2 - 2\sqrt{2}.a\alpha + a^2 = 0,$$
whence
$$\alpha = \frac{a}{2\sqrt{2}}.$$

The equation then becomes
$$y''^2 = \sqrt{2}.ax''.$$

Thus the latus-rectum of the parabola is equal to $a\sqrt{2}$, and the coordinates of the vertex, referred to the original axes of coordinates, are $\frac{1}{4}a$, $\frac{1}{4}a$.

The diagram will elucidate the analytical transformations.



$$\angle xOx' = \tfrac{1}{4}\pi,$$

$$OA = \frac{a}{2\sqrt{2}}.$$

$$OH = \alpha\cos\tfrac{1}{4}\pi = \tfrac{1}{4}a,$$
$$AH = \alpha\sin\tfrac{1}{4}\pi = \tfrac{1}{4}a.$$

$$AS = \frac{a}{2\sqrt{2}}.$$

2. To find the magnitude of the latus-rectum of the parabola of which the equation is

$$(y - mx)^2 = cx.$$

The required magnitude $= \dfrac{c}{(1 + m^2)^{\frac{3}{2}}}$.

3. To find the position and dimensions of the curve

$$x^2 + 2xy + y^2 - 3x + y + 1 = 0.$$

The latus-rectum $= \dfrac{1}{\sqrt{2}}$, the coordinates of the vertex are $\frac{5}{8}$, $-\frac{1}{8}$, and the inclination of the axis of figure to the axis of x is 45°.

4. To find the position of the parabola represented by the equation

$$(x + a)^{\frac{1}{2}} + (y + a)^{\frac{1}{2}} = 2a^{\frac{1}{2}},$$

and the magnitude of its latus-rectum.

Its vertex is at the origin of coordinates, its axis bisects the angle between the axes of x and y, and its latus-rectum is equal to $4a\sqrt{2}$.

5. The equation to a parabola being

$$(x + y\sqrt{3})^2 - 8x(1 + \sqrt{3}) + 8y(1 - \sqrt{3}) + 48 = 0,$$

to find its latus-rectum and the inclination of its axis to the axis of x.

The latus-rectum $= 4$, and the required inclination is equal to $\frac{1}{6}\pi$.

SECTION X.

Polar Equation. Focus the Pole.

1. If, with the focus S of a parabola as centre, a circle be described, passing through the vertex, to prove that the rectangle under the intercepts PQ, qp, of any focal chord $PQSqp$, included between the circle and the parabola, is constant.

The polar equation to the parabola being

$$r = m \sec^2 \tfrac{1}{2}\theta,$$

and that to the circle $r = m$,

the difference between these radii vectores for any value of θ is equal to

$$m \tan^2 \tfrac{1}{2}\theta.$$

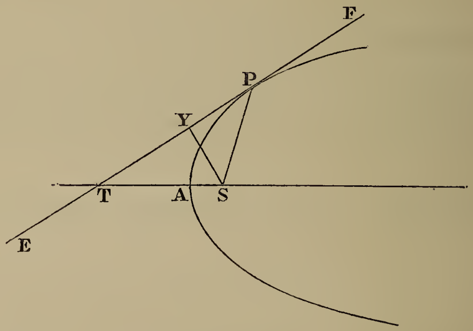
The corresponding difference for a value $\pi + \theta$ of the angular coordinate is equal to

$$m \cot^2 \tfrac{1}{2}\theta.$$

The quantities $m \tan^2 \tfrac{1}{2}\theta$, $m \cot^2 \tfrac{1}{2}\theta$, are the values of the intercepts PQ, pq: the area of the rectangle included by them is equal to the constant quantity m^2.

2. To find the locus of the vertex of a series of parabolas which have a given focus and touch a given line.

Let EF be the given line, and S the given focus. Draw SY at right angles to EF. Let P be the point at which one of the parabolas, of which A is the vertex, is touched by EF. Let EF cut the axis of the parabola PA in T.



Let $\angle ASY = \theta$, $\angle ASP = \phi$, $YS = c$, $AS = m$.

Then, by the nature of parabolas,

$$c = ST.\cos\theta = SP\cos\theta.$$

But $$SP = \frac{2m}{1 + \cos\phi}:$$

hence $$c = \frac{2m\cos\theta}{1 + \cos\phi}$$

$$= \frac{2m\cos\theta}{1 + \cos 2\theta}$$

$$= \frac{m}{\cos\theta},$$

or $$c\cos\theta = m,$$

which shews that the locus of A is a circle of which SY is the diameter.

This proposition is easily proved geometrically. We know, by a property of the parabola, that $\angle SAY$ is a right angle; hence the locus of A is a circle on SY as diameter.

Lardner: *Algebraic Geometry*, p. 129.

3. If r, r', be two radii vectores of a parabola at right angles to each other, and l the semi-latus-rectum, to prove that

$$\left(\frac{1}{r} - \frac{1}{l}\right)^2 + \left(\frac{1}{r'} - \frac{1}{l}\right)^2 = \frac{1}{l^2}.$$

4. In the focal distance SP of any point P in a parabola, Sp is taken equal to the distance PN of P from the axis; to find the equation to the locus of p.

The equation to the parabola being

$$r = \frac{2m}{1 + \cos\theta},$$

that to the required locus will be

$$r = 2m\tan\tfrac{1}{2}\theta.$$

5. If, from any point P in a parabola, a perpendicular PK be drawn to the directrix, to prove that, S being the focus, SP, SK, and the latus-rectum, are in geometrical progression.

6. If A be the vertex, S the focus, and PSp a focal chord of a parabola, to prove that the rectilineal triangle PAp varies as $(Pp)^{\frac{1}{2}}$.

7. From any point in the directrix of a parabola, two tangents are drawn to meet the curve: to determine the length of the latus-rectum of the parabola, the lengths of the two tangents being given.

If a, b, be the lengths of the two tangents, the length of the latus-rectum will be equal to

$$\frac{4a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}.$$

8. A chord QSQ' is drawn through the focus S of a parabola, parallel to the tangent at a point P; to prove that, L denoting the latus-rectum,

$$SQ.SQ' = L.SP.$$

9. If P be a point in the radius vector produced of a parabola $r(1+\cos\theta) = 2m$, such that its distance from the focus is equal to the corresponding focal chord; to find the equation to the locus of P, and to shew that in this curve, if two radii vectores be taken at right angles to each other, the sum of their reciprocals is equal to the reciprocal of the latus-rectum of the given parabola.

The equation to the required locus is

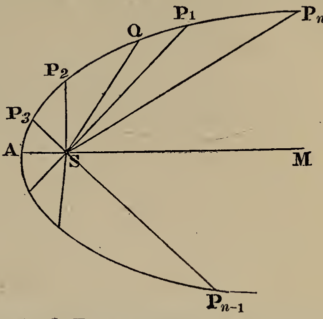
$$r = \frac{4m}{\sin^2\theta}.$$

10. If PT, pt, be the tangents, and PG, pg, the normals at P, p, the extremities of any focal chord PSp in a parabola, T, t, being the intersections of the tangents and G, g, of the normals, with the axis, to prove that

$$\frac{PT.PG}{pt.pg} = \tan^4\frac{ASP}{2}.$$

11. Let S be the focus of a parabola $P_1P_2P_3 \ldots P_n$, of which the vertex is A, axis AM, and latus-rectum L. Draw any line SP_1, and make n angles P_1SP_2, P_2SP_3, …… P_nSP_1, round S, all equal to each other. Draw SQ so that the angle $MSQ = n$ times the angle MSP_1. It is required to prove that

$$SP_1 . SP_2 . SP_3 \ldots\ldots SP_n = L^{n-1} . SQ.$$



Herschel: *Leybourn's Mathematical Repository, New Series*, vol. IV. p. 67.

SECTION XI.

Polar Equation. Vertex the Pole.

1. Two given parabolas have a common vertex A and a common axis: P, P', are points in the two curves, such that AP, AP', are at right angles and equal to each other: to find the magnitude and positions of AP, AP'.

Let c, c', be the two latera recta; let AP be inclined to the axis at an angle θ, and let r denote its magnitude. Then, by the polar equation to the curve AP,

$$r \sin^2 \theta = c \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, by the polar equation to the curve AP',

$$r \cos^2 \theta = c' \sin \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) there is

$$\tan \theta = \left(\frac{c}{c'}\right)^{\frac{1}{3}};$$

which determines the position of AP, AP'.

Also, from (1) and (2),

$$r = c \cos \theta + c' \sin \theta$$

$$= \frac{c c'^{\frac{1}{3}} + c' c^{\frac{1}{3}}}{\left(c^{\frac{2}{3}} + c'^{\frac{2}{3}}\right)^{\frac{1}{2}}} = (cc')^{\frac{1}{3}} . \left(c^{\frac{2}{3}} + c'^{\frac{2}{3}}\right)^{\frac{1}{2}},$$

which determines the magnitude of AP or AP'.

2. Having given r, r', two radii vectores of a parabola at right angles to each other, the vertex being the pole and the axis the prime radius vector, to find the latus-rectum.

If l denote the latus-rectum,

$$l^2 = \frac{(rr')^{\frac{4}{3}}}{r^{\frac{2}{3}} + r'^{\frac{2}{3}}}.$$

SECTION XII.

Polar Equation. Pole a point in the Axis.

1. A point C is taken in the axis of a parabola, at a distance from the vertex A greater than half the latus-rectum; to determine the least distance CP of this point from the curve.

Let $CA = c$, and $4m =$ the latus-rectum. Then, C being taken as the pole, the polar equation to the curve, if we put $\angle PCA = \theta$, $CP = r$, will be

$$r^2 \sin^2\theta = 4m(c - r\cos\theta),$$

whence
$$r^2\cos^2\theta - 4mr\cos\theta = r^2 - 4mc,$$
$$(r\cos\theta - 2m)^2 = r^2 - 4m(c - m).$$

From this result it is evident that $\{4m(c-m)\}^{\frac{1}{2}}$ is the least value of r. The corresponding value of θ is given by the equation

$$\cos\theta = \frac{2m}{r} = \left(\frac{m}{c-m}\right)^{\frac{1}{2}};$$

the value of $\cos\theta$ being always possible, because c is greater than $2m$, and therefore $c - m$ than m.

If the ordinate drawn from P meets the axis in M, then, from the value $\frac{2m}{r}$ of $\cos\theta$, we see that CM is equal to half the latus-rectum.

Apollonius: *Conicorum Liber Quintus*, prop. 8.

2. In the axis of a parabola is taken a point C, at a distance from its vertex A equal to half the latus-rectum. P is any

point in the curve; CP is joined. PM is an ordinate to the axis. To prove that CA is the least value of CP and that

$$(CP)^2 - (CA)^2 = (AM)^2.$$

Apollonius: *Conicorum*, lib. v. prop. 4.

SECTION XIII.

Polar Equation. Pole anywhere.

1. Through a given point within a parabola, a chord is drawn; to find the position of the chord that the rectangle of its two segments may be equal to a given square.

Let h, k, be the coordinates of the given point.

The equation to the parabola, referred to its axis and tangent, being

$$y^2 = lx,$$

the polar equation, (h, k) being the pole, will be

$$(r \sin\theta + k)^2 = l(r\cos\theta + h),$$

or $$r^2 \sin^2\theta + (2k\sin\theta - l\cos\theta)\, r + k^2 - lh = 0.$$

If r_1, r_2, be the roots of this equation,

$$r_1 . r_2 = \frac{k^2 - lh}{\sin^2\theta}.$$

The point (h, k) lying within the parabola, $k^2 - lh$ is negative: hence, c denoting the side of the given square, we shall have, equating c^2 to the area of the rectangle between the two segments of the chord,

$$\frac{lh - k^2}{\sin^2\theta} = c^2,$$

or $$\sin^2\theta = \frac{lh - k^2}{c^2}.$$

This equation gives two values of θ, or two positions of the chord; provided that c^2 is not less than $lh - k^2$, when the problem is impossible.

Cor. If the given point were without the parabola, we should have

$$\sin^2\theta = \frac{k^2 - lh}{c^2}.$$

De la Hire: *Sectiones Conicæ*, lib. v. prop. 39.

2. Through a given point O, within a parabolic area, is drawn a chord POQ: to find the inclination of POQ to the axis in order that the rectangle $PO.OQ$ may be the least possible.

The chord must be at right angles to the axis.

De la Hire: *Sectiones Conicæ*, lib. vii. prop. 28.

3. If, from a point T without a parabola, two tangents be drawn to the curve, meeting it in P and Q, and the axis in E and F respectively; to prove that

$$\frac{PT}{QT} = \frac{ET}{FT}.$$

Section XIV.

Linear Equation.

1. To prove that, in a parabola, the semi-sum of the focal radii vectores which terminate at the extremities of any arc of the curve, is always equal to the radius vector which terminates at the end of the diameter drawn through the middle point of the chord, together with the part of this diameter intercepted between the arc and the chord.

Let x, x', be the abscissæ of the ends of the arc, and $x_{,}$ the abscissa of the end of the diameter. Let r, r', $r_{,,}$ be the corresponding radii vectores. Let also $4m$ be the latus-rectum. Then, by the linear equation to the parabola, we have

$$r = m + x,$$

$$r' = m + x',$$

and therefore $$\tfrac{1}{2}(r + r') = m + \tfrac{1}{2}(x + x').$$

But $\frac{1}{2}(x + x')$ is equal to the abscissa of the middle point of the chord, that is, to $x_{,} + c$, where c represents the intercepted part of the diameter: hence

$$\tfrac{1}{2}(r + r') = m + x_{,} + c$$
$$= r_{,} + c.$$

2. The abscissæ of two points in a parabola, reckoned along the axis, are x, $3x$, and the corresponding focal distances r, $2r$: to find the position of the former of these points.

The required point is an extremity of the latus-rectum.

SECTION XV.

Polar Equation to the Tangent.

1. If PT, QT, be tangents at the points P, Q, of a parabola, the focus of which is S, to prove that

$$SP.SQ = ST^2.$$

A being the vertex of the parabola, let $\angle ASP = \alpha$, and $AS = m$. Then the polar equation to PT will be

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \alpha) \ldots\ldots\ldots\ldots\ldots\ldots (1).^*$$

Similarly, if $\angle ASQ = \beta$, the polar equation to QT will be

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \beta) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

At the intersection T, of (1) and (2),

$$\cos(\theta - \alpha) = \cos(\theta - \beta),$$

whence $\quad \theta - \alpha = -(\theta - \beta), \quad \theta = \tfrac{1}{2}(\alpha + \beta),$

and therefore

$$\frac{2m}{ST} = \cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2} = 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2},$$

$$ST = \frac{m}{\cos\frac{1}{2}\alpha . \cos\frac{1}{2}\beta}.$$

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[*] This equation to the tangent of a parabola is given by Mr. Davies in the *Philosophical Magazine* for 1842, p. 191.

Whence, by the polar equation to the parabola,

$$ST^2 = \frac{m}{\cos^2 \frac{1}{2}\alpha} \cdot \frac{m}{\cos^2 \frac{1}{2}\beta}$$

$$= SP.SQ.$$

2. To prove that a circle, described about the triangle formed by three tangents to a parabola, will pass through the focus.

The polar equation to a parabola being

$$r = \frac{2m}{1 + \cos\theta},$$

the polar equations to three tangents will be

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \alpha) \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \alpha') \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \alpha'') \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

α, α', α'', being the angular coordinates of the three points of contact.

If $\theta_{,,}$, $r_{,,}$, be the coordinates of the intersection of (1) and (2),

$$\theta_{,,} = \tfrac{1}{2}(\alpha + \alpha'), \qquad \frac{m}{r_{,,}} = \cos\frac{\alpha}{2} \,.\, \cos\frac{\alpha'}{2}.$$

Similarly, (θ, r), $(\theta_{,}, r_{,})$, being the coordinates of the intersections of (2), (3); (3), (1); we shall have

$$\theta = \tfrac{1}{2}(\alpha' + \alpha''), \qquad \frac{m}{r} = \cos\frac{\alpha'}{2}\cos\frac{\alpha''}{2};$$

and
$$\theta_{,} = \tfrac{1}{2}(\alpha'' + \alpha), \qquad \frac{m}{r_{,}} = \cos\frac{\alpha''}{2}\cos\frac{\alpha}{2}.$$

It is evident from these results that the three points (θ, r), $(\theta_{,}, r_{,})$, $(\theta_{,,}, r_{,,})$, all lie in a circle of which the equation is

$$\frac{m}{r}\cos\left(\theta - \frac{\alpha + \alpha' + \alpha''}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\alpha'}{2}\cos\frac{\alpha''}{2}.$$

The equation to the circle shews that it passes through the focus of the parabola.

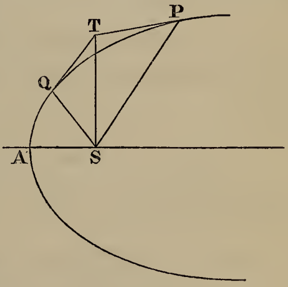
Poncelet: *Gergonne, Annales de Mathématiques*, tom. VIII. p. 9.
Davies, Goodwin: *Philosophical Magazine*, vol. XXI., 1842, pp. 190, 219.
Greathead: *Cambridge Mathematical Journal*, vol. I. p. 170.
Ellis: *Cambridge Mathematical Journal*, vol. I. p. 206.
Haydon: *Cambridge Mathematical Journal*, vol. IV. p. 192.

3. To determine the magnitude of the latus-rectum of a parabola the equation to which is

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1,$$

ω being the angle between the coordinate axes.

The form of the equation shews that the axes of x and y touch the parabola, a, b, being the distances of the points of contact from the origin.



Let PT, QT, touch the parabola in P, Q, respectively, PT being denoted by a, and QT by b; then $\angle PTQ = \omega$. Join SP, SQ, ST, S being the focus; let A be the vertex.

Put $\angle PSA = \alpha$, $\angle QSA = \beta$. Then the equations to PT, QT, are, respectively,

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \alpha), \quad \frac{2m}{r} = \cos\theta + \cos(\theta - \beta),$$

and therefore at T, the intersection of these two lines,

$$\theta = \tfrac{1}{2}(\alpha + \beta) = \angle AST,$$

whence $$\angle PST = \tfrac{1}{2}(\alpha - \beta) = \angle QST.$$

But $$\angle SPT = \tfrac{1}{2}(\pi - \alpha), \quad \angle SQT = \tfrac{1}{2}(\pi + \beta).$$

Hence $$\angle PTS = \tfrac{1}{2}(\pi + \beta), \quad \angle QTS = \tfrac{1}{2}(\pi - \alpha),$$

and consequently $$\omega = \angle PTQ = \pi - \tfrac{1}{2}(\alpha - \beta) \ldots\ldots\ldots\ldots (1).$$

From the triangle STP,

$$a = ST\frac{\sin\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\pi-\alpha}{2}}$$

$$= \frac{m}{\cos\frac{1}{2}\alpha.\cos\frac{1}{2}\beta}\cdot\frac{\sin\omega}{\cos\frac{1}{2}\alpha},$$

$$\frac{m\sin\omega}{a} = \cos^2\frac{\alpha}{2}.\cos\frac{\beta}{2}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Similarly, from the triangle STQ,

$$\frac{m\sin\omega}{b} = \cos^2\frac{\beta}{2}.\cos\frac{\alpha}{2}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3), we see that

$$\cos^3\frac{\alpha}{2} = \frac{mb}{a^2}.\sin\omega, \quad \cos^3\frac{\beta}{2} = \frac{ma}{b^2}.\sin\omega:$$

but, from (1),

$$\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta + \sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta = -\cos\omega,$$

$$\cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta + 2\cos\omega\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta + \cos^2\omega = \sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\beta$$

$$= 1 - \cos^2\tfrac{1}{2}\alpha - \cos^2\tfrac{1}{2}\beta + \cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta,$$

$$\cos^2\tfrac{1}{2}\alpha + 2\cos\omega\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta + \cos^2\tfrac{1}{2}\beta = \sin^2\omega;$$

hence, putting for $\cos\frac{1}{2}\alpha$, $\cos\frac{1}{2}\beta$, their values, we have

$$\frac{m^{\frac{2}{3}}}{(ab)^{\frac{4}{3}}}(a^2 + 2ab\cos\omega + b^2) = (\sin\omega)^{\frac{4}{3}},$$

and therefore

$$\text{latus-rectum} = \frac{4a^2b^2\sin^2\omega}{(a^2 + 2ab\cos\omega + b^2)^{\frac{3}{2}}}.$$

4. If a straight line be drawn from the focus of a parabola, intersecting at an angle β the tangent at any point of the curve; to find the equation to the locus of the intersection.

The equation to the parabola being

$$r = \frac{2m}{1+\cos\theta},$$

the required locus will be a straight line of which the equation is

$$r \sin \beta = \frac{m}{\sin(\theta + \beta)},$$

and which is therefore a tangent to the curve at a point of which the angular coordinate is $\pi - 2\beta$.

5. A right angle moves in such a manner that its sides always touch respectively two confocal parabolas, the axes of which lie in the same line; to find the locus of the summit of the angle.

The equations to the parabolas being

$$\frac{2m}{r} = 1 + \cos\theta, \quad \frac{2m'}{r} = 1 + \cos\theta,$$

the required locus will be a right line, perpendicular to the axis of the parabolas, defined by the equation

$$r \cos\theta = m + m'.$$

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XIX. p. 323.

Section XVI.

Poles and Polars.

1. If, from any point of the exterior of two similar parabolas, having the same axis, tangents be drawn to the interior one, to shew that they will touch it at the extremities of diameters, the distance between which is constant.

Let the equation to the interior parabola be

$$y^2 = 4mx \text{.......................} (1),$$

and to the exterior $\quad y^2 = 4m(x + c) \text{....................} (2).$

The equation to the polar in the parabola (1), corresponding to a pole (x_1, y_1), is

$$yy_1 = 2m(x + x_1)$$

or $\quad 2yy_1 = 4mx + 4mx_1 \text{ ....} (3).$

At the intersections of (1) and (3), we have

$$y^2 - 2y_1y + 4mx_1 = 0.$$

Let y', y'', be the ordinates of these intersections : then

$$y'' + y' = 2y_1, \qquad y''y' = 4mx_1,$$

$$(y'' - y')^2 = 4y_1^2 - 16mx_1 \dots\dots\dots\dots\dots (4).$$

Suppose the point (x_1, y_1), to lie in (2) : then, from (2) and (4),

$$(y'' - y')^2 = 16mc, \qquad y'' - y' = 4\,(mc)^{\frac{1}{2}},$$

which is the distance between the two diameters.

2. To find the locus of a pole, relatively to a given parabola, the portion of the corresponding polar intercepted by the curve being constant.

Let the equation to the parabola be

$$y^2 = 4mx \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Then, (h, k) being the pole, the equation to the polar will be

$$ky = 2m\,(x + h) \dots\dots\dots\dots\dots\dots\dots (2).$$

Let (x_1, y_1) be the middle point of the intercepted portion of the polar: then, putting in (1)

$$x = x_1 + r\cos\theta, \qquad y = y_1 + r\sin\theta,$$

we shall have

$$r^2\sin^2\theta + 2r\,(y_1\sin\theta - 2m\cos\theta) + y_1^2 - 4mx_1 = 0 \dots (3).$$

Also (2) becomes

$$k\,(r\sin\theta + y_1) = 2m\,(r\cos\theta + x_1 + h);$$

but, (x_1, y_1) being a point in (2),

$$ky_1 = 2m\,(x_1 + h) \dots\dots\dots\dots\dots\dots\dots (4),$$

and therefore

$$k\sin\theta = 2m\cos\theta \dots\dots\dots\dots\dots\dots\dots (5).$$

At the intersection of (3) and (5), θ is common to both, and the values of r in (3) are c and $-c$, $2c$ denoting the length of the intercepted portion of the polar.

Hence, by the theory of equations,

$$y_1\sin\theta = 2m\cos\theta \dots\dots\dots\dots\dots\dots\dots (6),$$

$$c^2\sin^2\theta = 4mx_1 - y_1^2 \dots\dots\dots\dots\dots\dots (7).$$

From (5) and (6) we see that

$$y_1 = k \quad\text{..............................}\quad (8).$$

From (4) and (7)

$$c^2 \sin^2\theta + y_1^2 = 2ky_1 - 4mh;$$

whence, by (8), $$c^2 \sin^2\theta = k^2 - 4mh,$$

or $$c^2 \sin^2\theta = (k^2 - 4mh)(\cos^2\theta + \sin^2\theta),$$

and therefore, by (5),

$$4m^2c^2 = (k^2 - 4mh)(k^2 + 4m^2),$$

the equation to the required locus.

3. If the normal at one extremity of the latus-rectum of a parabola be the polar, to prove that the corresponding pole will lie in the diameter through the other extremity of the latus-rectum, and to determine its exact position in this line.

If $y^2 = 4mx$ be the equation to the parabola, the coordinates of the pole will be

$$x = -3m, \qquad y = -2m.$$

4. To find the locus of the intersection of a perpendicular drawn from the vertex of a parabola to meet a polar, the pole of which lies in a similar external parabola having an axis coincident with that of the inner parabola.

If the equation to the inner parabola be

$$y^2 = 4mx,$$

then, a denoting the distance between the vertices of the two parabolas, the equation to the required locus will be

$$(m + x)y^2 = (a - x)x^2.$$

5. Any number of parabolas are described having the same vertex and axis, and any straight line is drawn without the parabolas at right angles to the common axis. If any points whatever in this line be taken as poles, to prove that all the polars belonging to all the parabolas will intersect in a single point.

6. The polar belonging to a pole P, relatively to a given parabola, intersects the curve in points E and F: the triangle EPF being supposed to have a constant area, to find the locus of P.

The equation to the parabola being

$$y^2 = lx,$$

and c^2 denoting the constant area, the required locus will be a parabola defined by the equation

$$y^2 = lx + (\tfrac{1}{2}c^2l)^{\frac{2}{3}}.$$

SECTION XVII.

Intersection of Parabolas.

1. If two parabolas have their axes perpendicular to each other; to prove that, if they cut each other in four points, those four points will lie in a circle.

The equations to the two parabolas, supposing the axes of coordinates to be parallel to their two axes, will be of the forms

$$y^2 + \beta y + \alpha x + \gamma = 0,$$
$$x^2 + \beta' x + \alpha' y + \gamma' = 0.$$

At the intersections of these parabolas, we shall have

$$x^2 + y^2 + (\alpha + \beta')\, x + (\beta + \alpha')\, y + \gamma + \gamma' = 0.$$

This equation shews that all their points of intersection lie in a circle.

2. Three parabolas, of which the axes are parallel to each other, intersect: to prove that the three chords joining their points of intersection pass through a single point.

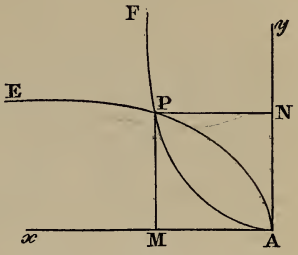
3. If a parabola, of which the equation is $y^2 = lx$, be intersected in four points by a parabola, the diameters of which are parallel to the axis of y, to prove that the sums of the ordinates of intersection on opposite sides of the axis of abscissæ, will be equal to each other.

De la Hire: *Sectiones Conicæ*, lib. v. prop. 30.

4. Ax is the axis of a parabola AE, Ay being the tangent at A. Also Ay is the axis of a parabola AF, Ax being a tangent to it at A. To prove that, E, F, being the latera-recta of the parabolas AE, AF, respectively,

$$E : PM :: PN : F,$$

P being the point of intersection of the two curves, and PM, PN, at right angles to Ax, Ay, respectively.



This is one of two methods of finding two mean proportionals between two given lines E and F, discovered by Menechmus, a geometrician of the School of Plato. His two solutions of the problem of the two means afford the earliest instances of the application of geometrical loci and conic sections to the solution of problems which cannot be solved by means of the rule and compass, called by the ancients *solid problems.*

The problem of the two means excited much interest among the ancient philosophers by reason of its intimate connection with the celebrated Delian problem of the duplication of the cube. Many solutions were accordingly discovered by various Greek geometricians; by Eudoxus, Plato, Hero, Philo, Apollonius, Diocles, Pappus, Sporus, Menechmus, Architas, Eratosthenes, and Nicomedes. The solution of Diocles, as depending upon the construction of his cissoid, has been rendered more complete by Newton, who has given in his *Universal Arithmetic* a method of describing this curve by continuous motion. For an exposition of the various solutions of this problem the reader is referred to the *Commentaries of Eutochius on the Second Book of Archimedes*, *Περὶ Σφαίρας καὶ Κυλίνδρου;* to the *Mathematicæ Collectiones* of Pappus, lib. III., where he will find an account of several of the solutions; and to Montucla's *Histoire des Mathématiques*, tom. I. p. 186.

Δηλίοις γὰρ λοιμώξασιν, ἔχρησεν ὁ Ἀπόλλων, ἀπαλλαγήσασθαι τοῦ λοιμοῦ, εἰ τὸν βωμὸν διπλασιάσουσιν, κυβικὸν ἔχοντα σχῆμα. οἵ δ᾽ ἐπῳκοδόμησαν προσθέντες τῷ προτέρῳ

βωμῷ ἕτερον κύβον ἴσον· ἀλλ' ἡ τῶν δύο κύβων συνθήκη, τὸ τοῦ κύβου σχῆμα ἠλλοίωσε· γέγονε δὲ ἀντὶ κύβου, δοκίς. τοῦ λοιμοῦ δὲ μὴ παυσαμένου, ἔχρησεν ὁ θεὸς μὴ πεποιηκέναι αὐτοὺς τὸ προσταχθέν. ὁ μὲν γὰρ προσέταξε διπλασιάσαι τὸν κύβον· τουτέστι, βωμὸν κατασκευάσαι κυβικὸν τοῦ προτέρου διπλάσιον· οἱ δέ, κύβον ἐπὶ κύβῳ ἐπέθηκαν. ἦλθον οὖν πρὸς Πλάτωνα *ζητοῦντες μέθοδον, πῶς ἂν τὸν κύβον διπλασιάσαιεν· ὁ δὲ πρὸς αὐτούς φησιν, ἔοικεν ὑμῖν ὀνειδίζειν ὁ θεὸς ὡς ἀμελοῦσι γεωμετρίας. ὁ δὲ τοῦ κύβου διπλασιασμὸς, εὑρεθήσεταί φησιν, εἰ δύο εὐθειῶν δύο μέσαι ἀνάλογον εὑρεθεῖεν· καὶ τοῦτο τὸ πρόβλημα τοῖς μαθηταῖς προεβάλλετο· οἵτινες καὶ περὶ τούτου γεγράφασιν, ὡς δεδύνηται ἕκαστος· ὧν οὐδέντι περισώζεται μέχρι τοῦ νῦν, ἀλλ' οὐδ' ὁ γεωμέτρης περὶ τούτου ἐπεσημήνατο· δύο μὲν γὰρ δοθεισῶν εὐθειῶν, μέσον ἀνάλογον εὑρεῖν, ἐξέθετο τὴν ἀπόδειξιν.*

'ΙΩΑΝΝΟΥ ΤΟΥ ΓΡΑΜΜΑΤΙΚΟΥ, ΕΙΣ ΤΑ ὝΣΤΕΡΑ 'ΑΝΑΛΥΤΙΚΑ 'ΑΡΙΣΤΟΤΕΛΟΥΣ, 'ΥΠΟΜΝΗΜΑ. *Venice*, 1534, p. 24.

Κομιζομένοις ἡμῖν ἀπ' Αἰ*γύπτου περὶ* Καρίαν Δηλίων *τινὲς ἀπήντησαν, δεόμενοι* Πλάτωνος *ὡς γεωμετρικοῦ λῦσαι χρησμὸν αὐτοῖς ἄτοπον ὑπὸ τοῦ θεοῦ προβεβλημένον. ἦν δὲ χρησμὸς,* Δη*λίοις καὶ τοῖς ἄλλοις* Ἕλλησι *παῦλαν τῶν παρόντων κακῶν ἔσεσθαι διπλασιάσασι τὸν ἐν* Δή*λῳ βωμόν· οὔτε δὲ τὴν διάνοιαν ἐκεῖνοι συμβάλλειν δυνάμενοι, καὶ περὶ τὴν τοῦ βωμοῦ κατασκευὴν γελοῖα πάσχοντες, (ἑκάστης γὰρ τῶν τεσσάρων πλευρῶν διπλασιαζομένης, ἔλαθον τῇ αὐξήσει τόπον στερεὸν ὀκταπλάσιον ἀπεργασάμενοι, δι' ἀπειρίαν ἀναλογίας, ἣ τῷ μήκει διπλάσιον παρέχεται)* Πλάτωνα *τῆς ἀπορίας ἐπεκαλοῦντο βοηθόν. ὁ δὲ, τοῦ* Αἰ*γυπτίου μνησθεὶς, προσπαίζειν ἔφη τὸν θεὸν* Ἕλλησιν, *ὀλιγωροῦσι παιδείας, οἷον ἐφυβρίζοντα τὴν ἀμαθίαν ἡμῶν, καὶ κελεύοντα γεωμετρίας ἅπτεσθαι μὴ παρέργου. οὐ γάρτοι φαῦλον, οὐδ' ἀμβλὺ διανοίας ὁρώσης, ἄκρως δὲ τὰς γραμμὰς ἠσκημένης ἔργον εἶναι, καὶ δυοῖν μέσων ἀνάλογον λῆψιν· ᾗ μόνη διπλασιάζεται σχῆμα κυβικοῦ σώματος, ἐκ πάσης ὁμοίως αὐξόμενον διαστάσεως. τοῦτο μὲν οὖν* Εὔδοξον *αὐτοῖς τὸν* Κνίδιον, *ἢ τὸν* Κυζικηνὸν Ἑλικῶνα, *συντελέσειν.*

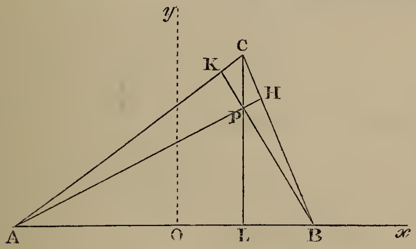
ΠΛΟΥΤΑΡΧΟΥ *Περὶ τοῦ* ΣΩΚΡΑΤΟΥΣ ΔΑΙΜΟΝΙΟΥ.

SECTION XVIII.

Parabolic Loci.

1. A triangle is constructed on a given base, its vertex lying in a given line parallel to the base: to find the locus of the intersection of perpendiculars from the extremities of the base on the opposite sides.

Let AB be the base, and C the vertex. The perpendiculars



AH, BK, CL, from A, B, C, on the opposite sides of the triangle, will meet in a single point P. Let Ox, Oy, be the axes of coordinates, O being the middle point of AB. Then, putting $AO = a = BO$, $CL = b$, $OL = x$, $PL = y$, we have

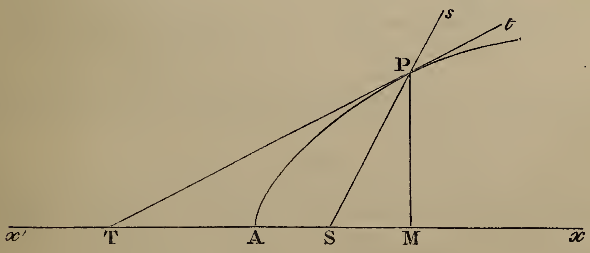
$$\frac{y}{a+x} = \tan \angle PAL = \cot \angle CBL = \frac{a-x}{b},$$

whence

$$a^2 - x^2 = by,$$

the equation to the locus of P, which is therefore a parabola, the latus-rectum of which is equal to the distance between AB and the locus of C, and which passes through A and B.

2. Two straight lines SPs, TPt, are drawn through two



points S, T, the former of which is fixed and the latter arbitrary, of an indefinite straight line xAx'. PM is at right angles to xx'. Supposing that the angles SPT, STP, are always equal, and that, A being a given point, AT is always equal to AM, to find the locus of P, in which the two straight lines intersect.

Let Ax be taken as the axis of x, and a line through A, at right angles to it, as that of y.

Then, denoting AS by m, the equations to Tt, Ss, will be respectively of the forms

$$y = \alpha x + \beta \text{..........................} (1),$$

$$y = \alpha' (x - m) \text{.......................} (2).$$

Since $\angle STP = \angle SPT$, we have

$$\alpha = \frac{\alpha' - \alpha}{1 + \alpha\alpha'} \text{ .... } (3);$$

and, since $AT = AM$, we have

$$\frac{\beta}{\alpha} = \frac{\beta + m\alpha'}{\alpha' - \alpha} \text{ ....} (4).$$

From (3) and (4) there is

$$\beta = \frac{\beta + m\alpha'}{1 + \alpha\alpha'},$$

$$\alpha\beta = m \text{ .... } (5).$$

From (4) we have $\quad 2\alpha\beta = \alpha' (\beta - m\alpha)$,

and therefore, by (5), $\quad 2m = \alpha' (\beta - m\alpha) \text{ .... } (6).$

From (2) and (6),

$$2m (x - m) = y (\beta - m\alpha) \text{ .... } (7).$$

From (1) and (7),

$$\alpha = \frac{y^2 - 2m (x - m)}{y (x + m)}, \qquad \beta = m \frac{y^2 + 2x (x - m)}{y (x + m)}.$$

Substituting these expressions for α and β in (5), and simplifying the result, we shall get

$$\{y^2 + (x - m)^2\} . (y^2 - 4mx) = 0.$$

Rejecting the former factor, the equating of which to zero would give impossible values to the coordinates, we have

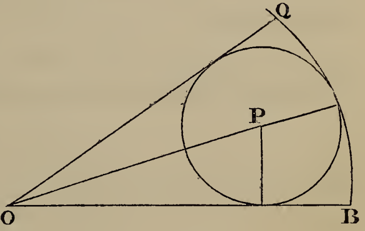
$$y^2 = 4mx,$$

the equation to a parabola of which A is the vertex and S the focus.

Raymond: *Gergonne, Annales de Mathématiques*, tom. III. p. 143.

3. To find the locus of the centre of a circle inscribed in a sector of a given circle, one of the bounding radii of the sector remaining fixed.

Let a be the radius of the circle of which OBQ is a sector, OB being the fixed radius. Let ρ denote the radius of the inscribed circle: r, θ, the polar coordinates of P, the centre of the inscribed circle, $\angle POB$ being equal to θ and OP to r: then, manifestly,



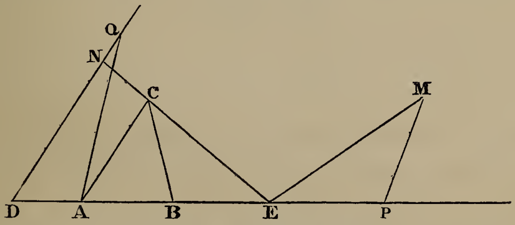
$$r + \rho = a,$$

and $$\rho = r \sin\theta:$$

hence $$r = \frac{a}{1 + \sin\theta},$$

the equation to a parabola of which O is the pole and focus, and of which the latus-rectum is equal to twice OB.

4. Two lines DE, DN, indefinitely produced, are given in position: MEN is a given angle, the summit E of which always moves along DE, while the side EN always passes through a given point C: moreover EM is a third proportional to NC, CE. To find the locus of M.



Draw CA parallel to ND; and draw CB so that $\angle CBE = \angle CEM$. From A draw AQ, meeting the indefinite line DN

in Q, so that $\angle BAQ = \angle ABC$. Draw MP parallel to QA, meeting the indefinite line DE in P.

Let $AD = a$, $AB = b$, $BC = c$, $AP = x$, $PM = y$, $AE = z$. Then, by the construction,

$$\angle EPM = \angle DAQ = \angle CBE:$$

also $$\angle PEM + \angle CEM = \angle CEP = \angle CBE + \angle BCE$$
$$= \angle CEM + \angle BCE,$$

and therefore $$\angle PEM = \angle BCE.$$

The two triangles PEM, BCE, are therefore equiangular.

By the condition of the problem,

$$(CE)^2 = EM.NC \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But, by the similar triangles PEM, BCE,

$$\frac{CE}{EM} = \frac{BC}{EP} = \frac{c}{x-z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, by the similar triangles CAE, NDE,

$$\frac{CE}{NC} = \frac{AE}{AD} = \frac{z}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1), (2), (3), $$\frac{z}{a} = \frac{x-z}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Again, by the similar triangles PEM, BCE,

$$\frac{PM}{PE} = \frac{BE}{BC},$$

or $$\frac{y}{x-z} = \frac{z-b}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Eliminating z between (4) and (5), we shall readily get, putting $a + c = f$, for the equation to the required locus,

$$x^2 - \frac{bf}{a}x - \frac{f^2}{a}y = 0.$$

"Le Comte Roger de Vintimille a proposé ce Problême avec quelques autres dans le Journal de Parme du mois d'Avril de l'année 1693, ce qui a donné occasion au Père Saquerius de faire imprimer un petit livre à Milan, dans lequel il avoue qu'il n' a pû resoudre celui-ci, quoiqu' il fasse assez paroître par la solution des autres qu' il est fort versé dans la Géométrie."

L'Hospital: *Traité Analytique des Sections Coniques*, p. 254.

5. To find the equation to the locus of a point the distance of which from the origin of coordinates is equal to its distance from the line

$$(y - c) \cos\theta + x \sin\theta = 0,$$

and to refer it to axes inclined to the original ones at an angle $\pi - \theta$.

The required locus is a parabola the equation to which, referred to the new axes, is

$$y^2 = c \cos\theta \, (c \cos\theta - 2x).$$

6. In a plane triangle ABC, if

$$\tan A \,.\, \tan \tfrac{1}{2} B = 2,$$

and AB be fixed; to find the locus of C.

If $AB = c$, $BC = r$, $\angle ABC = \theta$, the locus of C will be a parabola defined by the equation

$$r = \frac{c}{1 + \cos\theta}.$$

7. From any point Q, in a fixed ordinate of a parabola at right angles to its axis, a diameter is drawn to meet the curve in P. A being the vertex of the parabola, and AB the abscissa of Q, BA is produced to C, so that $AC = AB$. To find the locus of the intersection of AQ, CP, produced indefinitely.

The required locus is the parabola itself.

8. A is the origin, B a point in the axis of y, BQ a line parallel to the axis of x: in AQ, (produced if necessary), P is taken such that its ordinate is equal to BQ: to find the locus of P.

If $AB = c$, the locus will be a parabola defined by the equation

$$y^2 = cx.$$

9. If, from any point P of a circle, PC be drawn to the centre C, and a chord PQ be drawn parallel to the diameter ACB and bisected in R, to find the locus of the intersection of CP and AR.

If c be the radius of the circle; then, ABC being taken as the axis of x and a diameter, at right angles to it, as the axis of y, the required locus will be a parabola of which the equation is

$$y^2 = 2cx + c^2.$$

10. OA is a fixed straight line of length a, POP' a straight line through O making an angle θ with OA. To find the locus of P or P', having given that the product of the triangular areas AOP, AOP', is equal to a^4, and their quotient to $\cot^2 \frac{1}{2}\theta$.

The locus of P or P' will be a parabola represented by the equation

$$r = \frac{a}{1 + \cos\theta}.$$

11. Having given one side of a triangle, and the sum of the tangents of the adjacent angles, to find the locus of the vertex.

Taking the origin of rectangular coordinates at the middle point of the given side, the given side being chosen as the axis of x, we shall have, for the equation to the required locus,

$$m(a^2 - x^2) = 2ay,$$

$2a$ denoting the given side and m the given sum.

Lardner: *Algebraic Geometry*, p. 116.

12. To find the locus of the centre of a circle, which passes through a given point and touches a given straight line.

Let the given straight line be taken as the axis of x, and a line perpendicular to it, through the given point, as that of y. Let c be the ordinate of the given point. Then the required locus will be a parabola defined by the equation

$$x^2 + c^2 = 2cy.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 196, *troisième édition.*

Garnier: *Géométrie Analytique*, p. 180.

13. To find the locus of the centre of a circle which touches both a given circle and a given straight line.

Let the given straight line be chosen as the axis of x, and a perpendicular to it, through the centre of the given circle, as that of y. Then, γ being the radius of the given circle, and β the ordinate of its centre, the required locus will be a parabola defined by the equation

$$x^2 = (\beta \pm \gamma)(2y \pm \gamma - \beta).$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 197, *troisième édition.*
Garnier: *Géométrie Analytique*, p. 439, *deuxième édition.*

14. If SY be the perpendicular drawn from the focus of a given parabola upon the tangent at any point P; to find the locus of the centre of the circle circumscribing the triangle SYP.

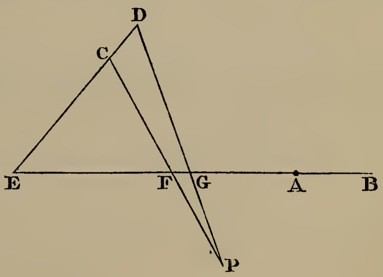
The equation to the given parabola being $y^2 = 4mx$, the required locus will be a parabola defined by the equation

$$y^2 = m(2x - m).$$

15. A straight line DCE, passing through two given points D, C, cuts a given straight line BA, produced, in E. Two points F, G, are taken in EA, such that

$$\frac{AF}{BG} = \frac{CE}{DE},$$

and the straight lines CF, DG, are produced to meet in P. To find the locus of P.



Let EAB, ECD, produced indefinitely, be taken as axes of x, y, respectively. Let $EC = k$, $ED = k'$, $EA = a$, $EB = a'$. Then the required locus will be a parabola defined by the equation

$$(ak' - a'k)(kk' + y^2) + kk'(k - k')x + \{a'k^2 - ak'^2 + (a' - a)kk'\}y = 0.$$

Leybourn: *Mathematical Repository, New Series*, vol. I. p. 45.

16. To find the locus of the focus of a parabola, which has a given vertex, and which touches a given right line.

Let A be the given vertex: from A draw AB at right angles to the given right line, meeting it in B. Let BA, produced indefinitely, be the axis of x, and Ay, at right angles to AB, be the axis of y.

Then, a denoting the distance AB, the required locus will be a parabola represented by the equation

$$y^2 = ax.$$

Lardner: *Algebraic Geometry*, p. 130.

17. Two straight lines OA, OB, are divided each of them into n equal parts. The ends of the r^{th} and $(r+1)^{\text{th}}$ divisions of OA, reckoning from A, are joined respectively to the ends of the r^{th} and $(r+1)^{\text{th}}$ divisions of OB, reckoning from O. If H be the middle point of AB, OHx be the axis of x, and Oy, parallel to BA, that of y; to prove that, if $OH = p$, $AH = q$, the locus of the intersection of the two joining lines will be a parabola defined by the equation

$$n^2py^2 - 2n^2q^2x + (n^2 - 1)\,pq^2 = 0.$$

Brianchon: *Journal de l' Ecole Polytechnique*, 19ᵉ *cahier*, tome XII.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 201, *troisième édition. Traité de Topographie*, *No.* 182, 2ᵉ *édition.*

Garnier: *Géométrie Analytique*, p. 443, *deuxième édition.*

SECTION XIX.

Parabolic Envelops.

1. To find the locus of the ultimate intersections of perpendiculars drawn to the normals of a parabola at the points where they cut the axis.

The abscissa of the intersection of a normal to a parabola $y^2 = 4mx$, drawn at a point (x, y) of the curve, with the axis, is $x + 2m$: the equation to the perpendicular to this normal will therefore be

$$y' = \frac{2m}{y}(x' - x - 2m),$$

or
$$yy' = 2m(x' - 2m) - \tfrac{1}{2}y^2.$$

Differentiating this equation with regard to the parameter y, we get

$$y' = -y:$$

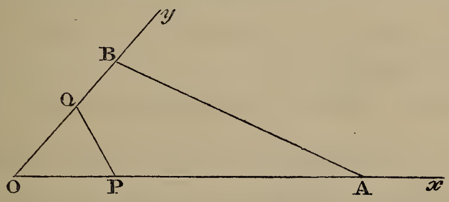
hence the equation to the required locus is

$$-y'^2 = 2m(x' - 2m) - \tfrac{1}{2}y'^2,$$

or
$$y'^2 = 4m(2m - x');$$

which is the equation to a parabola similar in form to the original one, its concavity being opposite and its axis coincident, and its vertex being at a distance $2m$ from the origin.

2. A straight line PQ moves with its extremities P, Q, in the sides OA, OB, of a triangle AOB, so that the rectangle of the segments OP, OQ, is always equal to that of the segments AP, BQ: to determine the curve to which PQ is always a tangent.



Let $OA = a$, $OB = b$, $OP = m$, $OQ = n$.

Then, OAx, OBy, being chosen as the axes of coordinates, the equation to PQ will be

$$\frac{x}{m} + \frac{y}{n} = 1 \text{ .... } (1),$$

m, n, being subject to the condition

$$mn = (a - m)(b - n),$$

or
$$\frac{m}{a} + \frac{n}{b} = 1 \text{ .... } (2).$$

Differentiating (1) and (2), we have

$$\frac{x}{m^2}\,dm + \frac{y}{n^2}\,dn = 0,$$

$$\frac{dm}{a} + \frac{dn}{b} = 0,$$

whence, λ being an arbitrary multiplier,

$$\frac{\lambda}{a} = \frac{x}{m^2}, \quad \frac{\lambda}{b} = \frac{y}{n^2},$$

whence $$\lambda\left(\frac{m}{a} + \frac{n}{b}\right) = \frac{x}{m} + \frac{y}{n},$$

or, by (1) and (2), $$\lambda = 1.$$

We have, therefore, $m^2 = ax$, $n^2 = by$,

and therefore, by (1) or (2),

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} \pm \left(\frac{y}{b}\right)^{\frac{1}{2}} = \pm 1,$$

which is the equation to a parabola to which OA, OB, are tangents at the points A, B.

Servois: *Gergonne, Annales de Mathématiques*, tom. IV. p. 156, 159.

3. Circles are described on successive double ordinates to the axis of a parabola as diameters: to shew that their envelop is an equal parabola, and that, of this system of circles, those of which the diameters are less than the latus-rectum, do not admit of an envelop.

4. If P be any point in a parabola, the equation of which is $y^2 = lx$, and M, N, be the points where the perpendiculars from P meet the axes of x, y, respectively; to find the locus of the continued intersection of lines MN.

The required locus is a parabola represented by the equation

$$y^2 = -4lx.$$

5. To find the locus of the consecutive intersections of straight lines drawn through points K, L, which move in

parallel straight lines so as to make OK always equal to OL, O being a fixed point in the locus of K.

Let O be the origin of rectangular coordinates, the locus of K being the axis of x. Let B be the intersection of the axis of y with the locus of L. Let $OB = b$. Then the required locus will be a parabola defined by the equation

$$x^2 = b.(b - 2y).$$

6. A straight line cuts off from a parabola a segment equal to a given area: to find the equation to the curve to which this line in any position is a tangent.

Let c^2 represent the given area; then, the equation to the parabola being

$$y^2 = 4mx,$$

that to the required envelop will be

$$y^2 = 4m\left\{x - \left(\frac{c^4}{4m}\right)^{\frac{1}{3}}\right\}.$$

SECTION XX.

Miscellaneous Problems.

1. The equation to a parabola, referred to rectangular axes, being

$$y^2 + 4ay \cot \alpha = 4ax,$$

to find its equation when referred to oblique axes inclined to each other at an angle α, the axis of x remaining the same.

The required equation is

$$y^2 = \frac{4a}{\sin^2 \alpha} . x.$$

2. P is any point in a parabola of which the vertex is A, and focus S; T is the point where the directrix intersects the axis; TP is joined and produced to cut the latus-rectum in N; SPQ is drawn to meet NQ, which is parallel to ST, in Q: to find the locus of Q.

The equation to the parabola being $y^2 = 4mx$, the required locus will be a circle represented by the equation

$$x^2 + y^2 = m(2x + 3m).$$

3. To find the locus of the intersection of the tangent at one extremity of a focal chord of a parabola with the ordinate at the other extremity produced.

The equation to the parabola being

$$y^2 = 4mx,$$

that to the required locus will be

$$mxy^2 = (x^2 + y^2)^2.$$

4. A circle is inscribed in a parabola, the distance of the centre of the circle from the vertex of the parabola being to the radius of the circle as $n^2 + 1$ to $n^2 - 1$. To find the positions of the points of contact.

The equation to the parabola being $y^2 = lx$, the abscissa of the points of contact is

$$\frac{ln}{(n \pm 1)^2}.$$

5. Having given a diameter of a parabola, and a tangent through the vertex of the diameter, to find the locus of the vertex of the curve.

The given diameter being taken as the axis of x and a perpendicular to it through its vertex, as that of y, the required locus is a straight line of which the equation is

$$y = 2x \tan \alpha,$$

where α denotes the angle between the given tangent and diameter.

Lardner: *Algebraic Geometry*, p. 132.

6. Under the conditions of the preceding problem, to find the locus of the focus.

The axes of coordinates being the same as in the preceding problem, the required locus is a straight line represented by the equation

$$y = x \tan 2\alpha.$$

Lardner: *Algebraic Geometry*, p. 132.

7. The tangent to a parabola, at a point P, meets the latus-rectum produced in L, and the directrix in M; to find the locus of the centre of the circle inscribed in the triangle SML, S being the focus.

The equation to the parabola being

$$r = m \sec^2 \tfrac{1}{2}\theta,$$

that to the required locus is

$$r = \tfrac{1}{2}m . \sec\theta . \sec^2 \tfrac{1}{2}\theta.$$

ELLIPSE.

SECTION I.

Referred to its Axes. Ordinates.

1. IF α, β, are the coordinates of the centre of a circle to radius γ, which cuts in four points an ellipse

$$a^2y^2 + b^2x^2 = a^2b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

to find the value of the product of the four ordinates of intersection.

The equation to the circle will be

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2.$$

Multiplying this equation by b^2 and subtracting the result from the equation to the ellipse, we have

$$(a^2 - b^2)\,y^2 + 2b^2\beta y - b^2\beta^2 + 2b^2\alpha x - b^2\alpha^2 = b^2\,(a^2 - \gamma^2) \ldots (2).$$

Multiplying (1) by $4b^2\alpha^2$ we have

$$4a^2b^2\alpha^2y^2 + 4b^4\alpha^2x^2 = 4a^2b^4\alpha^2,$$

and therefore, by (2),

$$4a^2b^2\alpha^2y^2 + \{b^2\,(a^2 + \alpha^2 + \beta^2 - \gamma^2) - 2b^2\beta y - (a^2 - b^2)\,y^2\}^2 = 4a^2b^4\alpha^2,$$

or $\quad (a^2 - b^2)^2\,y^4 + \ldots\ldots + b^4\,\{(a^2 + \alpha^2 + \beta^2 - \gamma^2)^2 - 4a^2\alpha^2\} = 0.$

Hence, by the theory of equations, y_1, y_2, y_3, y_4, being the four ordinates of intersection, we have

$$y_1y_2y_3y_4 = b^4 \, . \, \frac{(a^2 + \alpha^2 + \beta^2 - \gamma^2)^2 - 4a^2\alpha^2}{(a^2 - b^2)^2}.$$

2. If ordinates y_1, y_2, divide the major axis of an ellipse into two parts x_1, x_1', and x_2, x_2', respectively, to shew that,

l denoting the latus-rectum,

$$\frac{y_1^2}{x_1 . x_1'} = \frac{l}{2a},$$

and

$$\frac{y_1^2}{y_2^2} = \frac{x_1 . x_1'}{x_2 . x_2'}.$$

Ἐὰν ἐν κυλίνδρου τομῇ εὐθεῖαι ἀχθῶσιν ἐπὶ τὸν διάμετρον τεταγμένως· ἔσται τὰ ἀπ᾽ αὐτῶν τετράγωνα πρὸς μὲν τὰ περιεχόμενα χωρία ὑπὸ τῶν ἀπολαμβανομένων ὑπ᾽ αὐτῶν πρὸς τοῖς πέρασι τῆς πλαγίας τοῦ εἴδους πλευρᾶς, ὡς τοῦ εἴδους ἡ ὀρθία πλευρὰ πρὸς τὴν πλαγίαν· πρὸς ἑαυτὰ δὲ ὡς τὰ περιεχόμενα χωρία ὑπὸ τῶν, ὡς εἴρηται, ἀπολαμβανομένων εὐθειῶν.

ΣΕΡΗΝΟΥ ΑΝΤΙΝΣΕΩΣ ΦΙΛΟΣΟΦΟΥ Περὶ Κυλίνδρου Τομῆς. Πρότασις ιή.

3. A perpendicular AH is drawn to the semi-axis-major CA of an ellipse, through its extremity A, equal to half the latus-rectum; an ordinate MP to the axis cuts the straight line CH in Q. To prove that

$$(PM)^2 = 2 \text{ area } QMAH.$$

Apollonius: *Conicorum*, lib. V. prop. 1.

4. An ellipse and a circle having the same centre, the axis major of the ellipse is equal to three times the radius of the circle: to determine the points of intersection of the two curves, supposing the foci to cut the distance between the extremities of the diameter and axis in a given ratio; and to determine that ratio when the ellipse and circle just touch each other.

Let $AE = c$, $AS = \frac{c}{m}$, A being one vertex of the axis, S the nearer focus, E the nearest point of the circle. Then

$$x = \pm \frac{3c}{3m - 1} (4m^2 - 6m + 1)^{\frac{1}{2}};$$

and, when the ellipse and circle just touch each other,

$$m = \tfrac{1}{4}(3 + \sqrt{5}).$$

Leybourn's *Mathematical Repository*, *New Series*, vol. IV. p. 130.

5. About the centre of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is described a circle, with a radius equal to the ordinate: to find the locus of the intersection of this circle with the ordinate.

The required locus is an ellipse defined by the equation

$$\frac{a^2 + b^2}{a^2b^2} \cdot x^2 + \frac{y^2}{b^2} = 1.$$

Lardner: *Algebraic Geometry*, p. 151.

SECTION II.

Referred to its Axes. Tangents.

1. If (α, β), (α', β'), be the coordinates of two points in a diameter of an ellipse, and be subject to the condition

$$\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} = 1,$$

to find the equations to the tangents at the extremities of this diameter.

If x, y, be the point of contact,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ....} (1).$$

But, the points (α, β), and (α', β'), lying in the same line with the point (x, y),

$$\frac{x}{y} = \frac{\alpha}{\beta} = \frac{\alpha'}{\beta'};$$

and therefore $$\frac{x^2}{\alpha\alpha'} = \frac{y^2}{\beta\beta'}:$$

hence, from (1), $$x^2\left(\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2}\right) = \alpha\alpha',$$

and consequently, by the condition of the problem,

$$x = \pm (\alpha\alpha')^{\frac{1}{2}}:$$

similarly $$y = \pm (\beta\beta')^{\frac{1}{2}}.$$

Hence the equations to the tangents are

$$\frac{(\alpha\alpha')^{\frac{1}{2}}}{a^2} \cdot x' + \frac{(\beta\beta')^{\frac{1}{2}}}{b^2} \cdot y' = \pm 1.$$

2. To find the area of the parallelogram formed by tangents at the extremities of diameters $2r$, $2r'$, of an ellipse, the angle between these diameters being θ.

Let the semi-axes of the ellipse be a, b, and (x, y), (x', y'), the extremities of $2r$, $2r'$, respectively.

Let ϕ denote the angle between the tangents at their extremities. Then

$$\cos\phi = \frac{\dfrac{xx'}{a^4} + \dfrac{yy'}{b^4}}{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)^{\frac{1}{2}} \cdot \left(\dfrac{x'^2}{a^4} + \dfrac{y'^2}{b^4}\right)^{\frac{1}{2}}},$$

and therefore

$$\sin^2\phi = 1 - \cos^2\phi = \frac{(xy' - x'y)^2}{a^4b^4} \cdot \frac{1}{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)\left(\dfrac{x'^2}{a^4} + \dfrac{y'^2}{b^4}\right)}$$

$$= \frac{(xy' - x'y)^2}{a^4b^4} \cdot \delta^2\delta'^2,$$

δ, δ', being the perpendiculars from the origin upon the two tangents.

But, u denoting the area required, it is easily seen that

$$u = 4\,\frac{\delta\delta'}{\sin\phi};$$

hence, λ, λ', denoting the inclinations of r, r', to the major axis,

$$u = \frac{4a^2b^2}{xy' - x'y} = \frac{4a^2b^2}{rr'(\sin\lambda'\cos\lambda - \cos\lambda'\sin\lambda)}$$

$$= \frac{4a^2b^2}{rr'\sin(\lambda' - \lambda)}$$

$$= \frac{4a^2b^2}{rr'\sin\theta}.$$

3. In the tangent at any point P of an ellipse is taken a point T, such that the angle PCT, subtended by PT at the

centre of the ellipse, is always a right angle: to find the locus of the point T.

The coordinates of P being $a \cos\theta$, $b \sin\theta$, the equation to PT will be

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad\dots\dots\dots\dots\dots\dots(1).$$

Also, A being the extremity of the axis major,

$$\tan \angle PCA = \frac{b \sin\theta}{a \cos\theta}.$$

Hence the equation to CT, which is perpendicular to CP, is

$$\frac{\sin\theta}{\cos\theta} = -\frac{ax}{by} \quad\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

At the intersection T of the lines (1) and (2), the equation to the former of which may be written

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = (\cos^2\theta + \sin^2\theta)^{\frac{1}{2}},$$

we have
$$\frac{x}{a}\cdot by - \frac{y}{b}\cdot ax = (b^2y^2 + a^2x^2)^{\frac{1}{2}},$$

$$\left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 = \left(\frac{a}{b} - \frac{b}{a}\right)^2,$$

which is the equation to the required locus.

4. To find the equation to the tangent of the ellipse

$$x^2 + 2y^2 = 3,$$

at the extremity of the latus-rectum.

The required equation is

$$\frac{x}{\sqrt{6}} + \frac{y}{3} = 1.$$

5. If h, k, represent the intercepts of the axes of coordinates, made by any tangent of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

to prove that
$$\frac{a^2}{h^2} + \frac{b^2}{k^2} = 1.$$

6. A tangent at any point P of an ellipse meets the semi-axis-major, produced, in T, and the semi-axis-minor, produced, in t; from P are drawn PM, Pm, perpendicular to CT, Ct, respectively. To compare the areas of the triangles PMT, Pmt.

The coordinates of P being x, y,

$$\text{triangle } PMT : \text{triangle } Pmt :: a^4y^4 : b^4x^4.$$

7. The circumstances of the preceding problem remaining the same, to find the locus of a point P' in Tt, such that

$$\frac{P'T}{P't} = \frac{n}{m}.$$

The equation to the required locus is

$$\left(\frac{ma}{x}\right)^2 + \left(\frac{nb}{y}\right)^2 = (m+n)^2.$$

8. If, from a point without an ellipse, two tangents be drawn to the curve, and two straight lines to the foci; to prove that the angle contained by one tangent and the line drawn to one focus, will be equal to the angle contained by the other tangent and the line drawn to the other focus.

Leybourn: *Mathematical Repository*, *New Series*, vol. II. p. 40.

9. To find the locus of the intersection of an ordinate of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with the perpendicular drawn from the centre upon the tangent at the extremity of this ordinate.

The required locus will be another ellipse represented by the equation

$$\frac{x^2}{a^2} + \frac{b^2y^2}{a^4} = 1.$$

Lentheric, Vallès, Bobillier: *Gergonne*, *Annales de Mathématiques*, tom. XVII. p 377.

10. AA', BB', are the diagonals of a rhombus $ABA'B'$, which intersect in C. If α, β, be the semi-axes of any ellipse described with C as a centre and axes in directions AA', BB', so as to touch the sides of the rhombus; then α, β, are coordinates of a point in the ellipse having AA', BB', for its axes.

11. In two ellipses, concentric and similarly placed, are taken two points P, Q, the abscissæ of which are as their major axes; P', Q', are two other such points. To prove that, θ, θ', ϕ, ϕ', being the angles which the tangents at P, P', Q, Q', respectively, make with the major axes,

$$\frac{\tan\theta}{\tan\theta'} = \frac{\tan\phi}{\tan\phi'}.$$

SECTION III.

Referred to its Axes. Magical Equation to the Tangent.

The equation to an ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the straight line defined by the equation

$$y = \alpha x + (a^2\alpha^2 + b^2)^{\frac{1}{2}},$$

will be a tangent to the ellipse, varying in position with the value of α.

This equation may be seen in Leroy's *Géométrie* and in Waud's *Algebraical Geometry:* it seems however, together with analogous equations to the tangents of parabolas and hyperbolas, to have been first employed by Mr. A. Smith for the demonstration of various properties of conic sections. Owing to its great utility, particularly in problems of tangency which do not involve the consideration of the point of contact, it has been called the *Magical Equation* to the tangent.

Smith: *Cambridge Mathematical Journal*, vol. I. p. 9.

If l, m, be taken to represent the direction-cosines of the tangent, its equation may be expressed in the more symmetrical form

$$lx + my = (l^2a^2 + m^2b^2)^{\frac{1}{2}}.$$

1. Two straight lines, such that the product of the trigonometrical tangents of their inclinations to the axis of x is constant, touch an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1:$$

to find the equation to the locus of their intersections.

The equation which connects the coordinates of any point in a tangent to an ellipse with α, the trigonometrical tangent of the inclination of the tangent to the axis of x, is

$$(y - \alpha x)^2 = a^2\alpha^2 + b^2,$$

or

$$(x^2 - a^2)\,\alpha^2 - 2xy\alpha + y^2 - b^2 = 0.$$

If therefore x, y, be the coordinates of the intersection of the two tangents, the two values of α in this quadratic will be the trigonometrical tangents of the inclinations of the two tangents to the axis of x. Hence, m denoting the constant product, we have for the equation to the required locus

$$y^2 - b^2 = m\,(x^2 - a^2),$$

which represents an ellipse or an hyperbola accordingly as m is negative or positive.

Rochat: *Gergonne, Annales de Mathématiques*, tom. II. p. 225.

Garnier: *Géométrie Analytique*, p. 275, *deuxième édition.*

2. To find the locus of the intersection of a pair of tangents to an ellipse, at right angles to each other.

The equation to one tangent being

$$x \cos\alpha + y \sin\alpha = \pm\,(a^2\cos^2\alpha + b^2\sin^2\alpha)^{\frac{1}{2}},$$

that to the other will be

$$-\,x \sin\alpha + y \cos\alpha = \pm\,(a^2\sin^2\alpha + b^2\cos^2\alpha)^{\frac{1}{2}}:$$

adding together the squares of these two equations, we get

$$x^2 + y^2 = a^2 + b^2,$$

as the equation to the required locus, which is therefore a circle concentric with the ellipse, having $(a^2 + b^2)^{\frac{1}{2}}$ for its radius.

> Lamé: *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*, p. 77.
>
> Puissant: *Recueil de diverses propositions de Géométrie*, p. 214, *troisième édition.*

3. If p, p', are the distances of the foci of an ellipse from a tangent, to prove that

$$p.p' = b^2.$$

The equation to a tangent being

$$lx + my = (l^2a^2 + m^2b^2)^{\frac{1}{2}},$$

where l, m, are its direction-cosines; and the coordinates of the foci being $\{(a^2 - b^2)^{\frac{1}{2}}, 0\}$, and $\{-(a^2 - b^2)^{\frac{1}{2}}, 0\}$, we have

$$p = -(l^2a^2 + m^2b^2)^{\frac{1}{2}} + l(a^2 - b^2)^{\frac{1}{2}},$$

$$p' = -(l^2a^2 + m^2b^2)^{\frac{1}{2}} - l(a^2 - b^2)^{\frac{1}{2}};$$

whence

$$\begin{aligned} pp' &= l^2a^2 + m^2b^2 - l^2(a^2 - b^2) \\ &= (l^2 + m^2)\,b^2 \\ &= b^2. \end{aligned}$$

4. To find the locus of the middle points of chords in a circle which touch a concentric ellipse.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation to the tangent will be, l, m, being its direction-cosines,

$$lx + my = (l^2a^2 + m^2b^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let the equation to the circle be

$$x^2 + y^2 = c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

At the intersections of (1) and (2),

$$x^2 - 2l(l^2a^2 + m^2b^2)^{\frac{1}{2}}.x + l^2a^2 + m^2b^2 - m^2c^2 = 0.$$

Hence, $x_{,}$ denoting the abscissa of the middle point of the chord, and x', x'', the abscissæ of its extremities,

$$x_{,} = \tfrac{1}{2}(x' + x'') = l\,(l^2a^2 + m^2b^2)^{\frac{1}{2}}.$$

Similarly $$y_{,} = \tfrac{1}{2}(y' + y'') = m\,(l^2a^2 + m^2b^2)^{\frac{1}{2}}.$$

Hence $$x_{,}^2 + y_{,}^2 = l^2a^2 + m^2b^2.$$

But also $$mx_{,} = ly_{,},$$

and therefore, by virtue of the equation

$$l^2 + m^2 = 1,$$

we have $$l^2\,(x_{,}^2 + y_{,}^2) = x_{,}^2, \quad m^2\,(x_{,}^2 + y_{,}^2) = y_{,}^2.$$

Hence $$(x_{,}^2 + y_{,}^2)^2 = a^2x_{,}^2 + b^2y_{,}^2,$$

which is the equation to the required locus.

5. To find the locus of the summit of a moveable right angle, one side of which touches one, and the other side the other, of two confocal ellipses.

Let the equations to the ellipses be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

where a, b, a', b', will be subject to the condition

$$a^2 - b^2 = a'^2 - b'^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equations to the two tangents will be respectively

$$lx + my = (l^2a^2 + m^2b^2)^{\frac{1}{2}},$$
$$-\,mx + ly = (m^2a'^2 + l^2b'^2)^{\frac{1}{2}}.$$

At the intersection of these two tangents, we have, squaring and adding their equations,

$$(l^2 + m^2)\,(x^2 + y^2) = l^2\,(a^2 + b'^2) + m^2\,(a'^2 + b^2)$$
$$= (l^2 + m^2)\,(a^2 + b'^2), \text{ by } (1),$$

or $$= (l^2 + m^2)\,(a'^2 + b^2):$$

thus the locus of the summit of the angle is a concentric circle represented by either of the equations

$$a^2 + b'^2 = x^2 + y^2 = a'^2 + b^2.$$

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XIX. p. 317.

6. To find the equation to a tangent to an ellipse

$$3x^2 + y^2 = 3,$$

inclined at an angle of $\frac{1}{4}\pi$ to the axis of x.

The required equation is

$$y = x + 2.$$

7. To find the distance of the centre of an ellipse from a tangent inclined to the major axis at an angle ϕ.

$$\text{Distance} = a\,(1 - e^2 \cos^2 \phi)^{\frac{1}{2}}.$$

8. To find the distance of the focus of an ellipse from a tangent inclined at an angle ϕ to the major axis.

$$\text{Distance} = a\,\{e \sin \phi + (1 - e^2 \cos^2 \phi)^{\frac{1}{2}}\}.$$

9. The tangent of an ellipse is inclined to the major axis at an angle ϕ: to find the area of the triangle included between this tangent and the two axes.

$$\text{Required area} = \tfrac{1}{2}(a^2 \tan \phi + b^2 \cot \phi).$$

10. A tangent to an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is inclined to the axis of x at an angle ϕ; to find the product of the distances of the ends of the axis major from this tangent.

The required product is equal to

$$b^2 \cos^2 \phi.$$

11. To find the equation to the locus of the intersection of a tangent to an ellipse with a perpendicular upon it from a given point.

The coordinates of the given point being α, β, and the equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the required locus will be defined by the equation

$$\{x(x-\alpha)+y(y-\beta)\}^2=a^2(x-\alpha)^2+b^2(y-\beta)^2.$$

If the given point be the centre, the equation becomes

$$(x^2+y^2)^2=a^2x^2+b^2y^2,$$

a result given by Vallès: *Gergonne, Annales de Mathématiques*, tom. XVII. p. 379.

12. To find the equation to the curve, from any point of which if two tangents be drawn to a given ellipse, the angle contained between them shall be constant.

If $\frac{m}{n}$ denote the value of the tangent of the constant angle, the required equation to the curve will be

$$m^2(x^2+y^2-a^2-b^2)^2=4n^2(a^2y^2+b^2x^2-a^2b^2).$$

13. If the two sides of a moveable right angle are always tangents to a given ellipse, its summit will describe a circle concentric with the ellipse, the radius of which is equal to the chord joining extremities of the major and minor axes.

Rochat: *Gergonne, Annales de Mathématiques*, tom. II. p. 228.

14. If two straight lines, always touching a given ellipse, move in such a way that the product of the trigonometrical tangents of the angles which they form with one of the axes be constant and negative, the point of intersection of the two tangents will describe a second ellipse. If we conceive two tangents to this second ellipse, moveable like the first, and subjected to the same conditions, the intersection of these last will describe a third ellipse from which, following the same course, we may deduce a fourth, and so on successively. Then

(1). The areas of these ellipses will form a geometrical progression having 2 for its common ratio.

(2). The tangents, the intersection of which shall describe any one of these ellipses, will be always parallel to two supplementary chords of the ellipse which shall immediately precede in the order of their successive generation.

Rochat: *Gergonne, Annales de Mathématiques*, tom. II. p. 228.

15. The semi-axes of an ellipse being a, b; to find the area of a parallelogram which circumscribes it, the inclinations of its sides to the major axis being given; and to determine the least of all such parallelograms.

If $\tan^{-1} m$ and $\tan^{-1} m'$ be the given inclinations, the required area will be equal to

$$4\left\{a^2b^2 + \left(\frac{a^2\,mm' + b^2}{m - m'}\right)^2\right\}^{\frac{1}{2}},$$

which is evidently the least possible when

$$a^2\,mm' + b^2 = 0,$$

or when the parallelogram is a conjugate parallelogram.

Durrande: *Gergonne, Annales de Mathématiques*, tom. XII. p. 142.

16. If parallelograms, which circumscribe an ellipse, have their areas constantly equal to k times that on the major and minor axes; to find the equation to the loci of their angular points.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the angular points lie in two ellipses represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2k\{1 \pm (k^2 - 1)^{\frac{1}{2}}\}.$$

SECTION IV.

Referred to its Axes. Normals.

1. A normal at a point P of an ellipse, cuts the major axis in G. To prove that GP is the shortest line which can be drawn from G to the curve.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

let h be the abscissa of P; then, C being the centre,

$$CG = e^2h.$$

Let r be the distance of G from any other point P' in the curve; then, x, y, being the coordinates of P',

$$\begin{aligned} r^2 &= (x - e^2h)^2 + y^2 \\ &= (x - e^2h)^2 + (1 - e^2)(a^2 - x^2) \\ &= e^2(x - h)^2 + (1 - e^2)a^2 - e^2(1 - e^2)h^2 \\ &= e^2(x - h)^2 + (1 - e^2)(a^2 - e^2h^2), \end{aligned}$$

and therefore r is the least possible when $x = h$, or when P' coincides with P.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 13.

2. P is a point in an ellipse, D a point in the major axis, such that PD is equal to the minor semi-axis: PQ is a normal at P meeting in Q a perpendicular to the major axis through D. To find the locus of Q.

Let (x', y') be the coordinates of P: then, the equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

that to the normal at P will be

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = a^2 - b^2 \qquad\text{......................(1).}$$

Again, since PD is equal to b, we have, x denoting the abscissa of D,

$$(x' - x)^2 + y'^2 = b^2,$$

and therefore, by the equation to the ellipse,

$$(x' - x)^2 + \frac{b^2}{a^2}(a^2 - x'^2) = b^2,$$

$$(x' - x)^2 = \frac{b^2}{a^2}x'^2,$$

$$x' - x = \mp\frac{b}{a}x',$$

$$x' = \frac{ax}{a \pm b} \qquad\text{......................(2).}$$

Substituting this value of x' in (1), we have, x, y, denoting the coordinates of Q,

$$\frac{b^2y}{y'} = b^2 - a^2 + a(a \pm b)$$

$$= b(b \pm a),$$

$$y' = \frac{by}{b \pm a} \quad \text{.................................} \quad (3).$$

Substituting the expressions for x', y', given in (2) and (3), in the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

we have, for the equation to the required locus,

$$x^2 + y^2 = (a \pm b)^2,$$

which represents either of two circles, concentric with the ellipse, and having $a - b$, $a + b$, as radii.

3. To find the equation to the normal at a point of an ellipse, defined by the equation

$$2x^2 + 3y^2 = 4,$$

the abscissa of the point being 1.

The required equation is

$$y = \pm \frac{1}{\sqrt{6}} (3x - 1).$$

4. To find the equation to the normal of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in terms of α, the inclination of the normal to the axis of x.

Putting $\cos\alpha = l$, $\sin\alpha = m$, we shall obtain, for the required equation,

$$\frac{x}{l} - \frac{y}{m} = \frac{a^2 - b^2}{(l^2a^2 + m^2b^2)^{\frac{1}{2}}}.$$

5. To find the length of that portion of the normal to an ellipse, which is included between the curve and the major axis, in terms of the inclination of the normal to this axis.

If θ denote the inclination of the normal, and $2a$, $2b$, the major and minor axes of the ellipse, the length of the normal will be equal to

$$\frac{b^2}{(a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}}.$$

6. To find the tangent of the angle between the normal at any point (x, y) of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the straight line drawn from this point to the centre of the ellipse.

The required tangent is equal to

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right) xy.$$

7. To find the length of the longer normal, drawn from a point in the minor axis of an ellipse at a given distance from the centre, and intercepted between that point and the curve.

If c denote the given distance, a the semi-axis major, and e the eccentricity, the length of the normal will be equal to

$$\left(a^2 + \frac{c^2}{e^2}\right)^{\frac{1}{2}}.$$

8. If a right angle moves so that its sides are constantly normals to an ellipse, to find the locus of the summit of the angle.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

that to the required locus will be

$$(a^2 + b^2) \,.\, (x^2 + y^2) \,.\, (a^2y^2 + b^2x^2)^2 = (a^2 - b^2)^2 \,.\, (a^2y^2 - b^2x^2)^2.$$

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVII. p. 277.

SECTION V.

Referred to its Axes. Chords.

1. Chords are drawn from the extremities of the transverse axis of an ellipse to any point in the curve, and perpendiculars, drawn to these chords from the same point, are produced to cut the same axis: to find the part of the axis which is intercepted between the perpendiculars.

The principal axes of the ellipse being taken as axes of coordinates, the equation to one of the chords will be, if x, y, be the coordinates of the point,

$$y' = \frac{y}{x-a}(x'-a),$$

and that to its perpendicular

$$y' - y = -\frac{x-a}{y}(x'-x).$$

Putting $y' = 0$ in the last equation, we have, for the intercept of the perpendicular on the axis of x,

$$x' = x + \frac{y^2}{x-a}$$

$$= x - \frac{b^2}{a^2}(a+x).$$

Similarly, the intercept of the other perpendicular, $-a$ being substituted for a, is equal to

$$x'' = x - \frac{b^2}{a^2}(-a+x).$$

Hence the part of the axis intercepted between the two perpendiculars

$$= x'' - x' = \frac{2b^2}{a}$$

$$= \text{the latus-rectum.}$$

Ivory: *Leybourn's Mathematical Repository, New Series*, vol. II. p. 175.

2. If, from any point in the circumference of an ellipse, two chords be drawn so as to touch the circle described about the minor axis as diameter, and if the chord joining the points of contact be produced to meet the major and minor axes in H, K, respectively, to prove that, a, b, being the semi-axes of the ellipse,

$$\frac{b^2}{(CH)^2} + \frac{a^2}{(CK)^2} = \frac{a^2}{b^2}.$$

Let (h, k) be the coordinates of any point in the ellipse from which the two chords are drawn. Then the equation to the chord of contact will be

$$hx + ky = b^2,$$

and therefore $$CH = \frac{b^2}{h}, \quad CK = \frac{b^2}{k}:$$

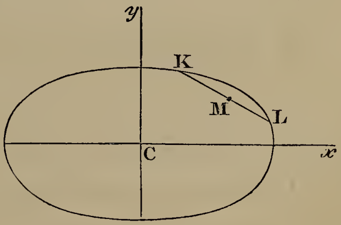
but, by the equation to the ellipse,

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1;$$

hence $$\frac{b^2}{(CH)^2} + \frac{a^2}{(CK)^2} = \frac{a^2}{b^2}.$$

3. To find the equation to the locus of the middle points of all chords of a given length in an ellipse.

Let M be the middle point of any chord KL. Let x', y', be the coordinates of L, and x, y, those of M. Also let $MK = r = ML$. Then, θ denoting the inclination of KL to the axis of x,



$$x' = x + r\cos\theta,$$
$$y' = y - r\sin\theta;$$

and therefore, by the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

we have

$$\frac{(x + r\cos\theta)^2}{a^2} + \frac{(y - r\sin\theta)^2}{b^2} = 1.$$

But, if $KL = 2c$, the two values of r in this equation must be each equal to c in magnitude and must have opposite signs: hence

$$\frac{x\cos\theta}{a^2} = \frac{y\sin\theta}{b^2},$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2\,\frac{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}{\cos^2\theta + \sin^2\theta} = 1.$$

From these two equations we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2\,\frac{a^2y^2 + b^2x^2}{a^4y^2 + b^4x^2} = 1;$$

which is the equation to the required locus.

4. P is any point in an ellipse, AA' its axis major, MP an ordinate to the point P: to any point Q in the curve are drawn the chords AQ, $A'Q$, meeting MP in R, S, respectively. To prove that

$$MR.MS = MP^2.$$

5. If, from a point P in the tangent at the extremity B of the axis minor of an ellipse, a tangent PQ be drawn, to prove that PQ will be equal to the chord BQ, if

$$BP : AC :: \sqrt{3} : 1.$$

6. If the equation to the chord of an ellipse be

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1,$$

to find the length of the chord.

If $2l$ be the length of the chord,

$$l^2 = (\alpha^2 + \beta^2)\,.\,a^2b^2.\,\frac{a^2\beta^2 + b^2\alpha^2 - \alpha^2\beta^2}{(a^2\beta^2 + b^2\alpha^2)^2}.$$

7. PR, QR, are two normals of an ellipse, cutting each other at right angles in R: CRH is a semi-diameter. To prove that the tangent at H is parallel to the chord PQ.

8. A chord of an ellipse, inclined at an angle $\tan^{-1}c$ to the axis major, passes through an extremity of the axis minor: if x, y, be the coordinates of the second point of the chord's intersection with the ellipse, to prove that, a, b, being the semi-axes,

$$\tan^{-1}\left(\frac{bx}{ay}\right) + 2\tan^{-1}\left(\frac{ac}{b}\right) = \pi.$$

9. To find the locus of the middle point of a chord of an ellipse, the equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

the chord being supposed to touch a concentric circle of radius r.

The equation to the required locus is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = r^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right).$$

10. If, through the extremity of either axis $2a$ of an ellipse, any two chords be drawn at right angles to each other, to prove that the chord joining the points in which the two chords cut the curve, will always pass through a point in the axis at a distance from its extremity equal to

$$a \cdot \frac{a^2 - b^2}{a^2 + b^2}.$$

11. To find the length of a chord of an ellipse, which is a tangent at any proposed point of a confocal ellipse.

Let x, y, be the coordinates of the point of contact, a, b, the semi-axes of the interior, and α, β, those of the exterior ellipse. Then, c denoting the required length,

$$\frac{c}{(\alpha^2 - a^2)^{\frac{1}{2}}} = 2\alpha\beta \cdot \frac{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}}{\dfrac{\alpha^2 x^2}{a^4} + \dfrac{\beta^2 y^2}{b^4}} = \frac{c}{(\beta^2 - b^2)^{\frac{1}{2}}}.$$

12. PQ, PR, are chords of an ellipse, drawn through any proposed point P in the curve, at right angles to each other:

to find the equation to the chord QR, to prove that QR always passes through a fixed point I, and, P being afterwards regarded as variable in position, to find the locus of the point I.

If $m, -\dfrac{1}{m}$, denote the tangents of the inclinations of PQ, PR, to the axis of x; then, $a\cos\theta$, $b\sin\theta$, being the coordinates of P, the equation to the chord QR will be

$$ab\left(\frac{1}{m} - m\right).(ay\cos\theta + bx\sin\theta) + (a^2 + b^2)(ay\sin\theta - bx\cos\theta) + ab(a^2 - b^2) = 0:$$

the point, through which this chord always passes, while m varies, is the intersection of the two lines

$$ay\cos\theta + bx\sin\theta = 0,$$

$$(a^2 + b^2).(ay\sin\theta - bx\cos\theta) + ab(a^2 - b^2) = 0,$$

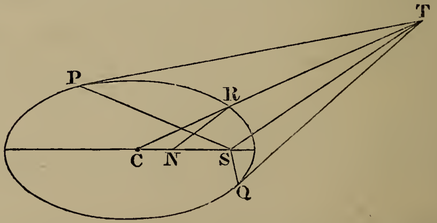
and, θ being now considered variable, the locus of this point is an ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2.$$

SECTION VI.

Referred to its Axes. Focal Properties.

1. PT, QT, are tangents to an ellipse at P, Q; S is one of the foci, C the centre, R the point where CT cuts the curve;



RN, parallel to ST, meets the major axis in N: to shew that

$$SP.SQ = RN^2.$$

Let x', x'', be the abscissæ of the two points of contact. Then

$$SP.SQ = (a - ex').(a - ex'').$$

The values of the abscissæ, if $x_{\prime}$, $y_{\prime}$, be the coordinates of T, will be obtained from the two equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{xx_{\prime}}{a^2} + \frac{yy_{\prime}}{b^2} = 1.$$

Eliminating y, we have

$$\left(1 - \frac{xx_{\prime}}{a^2}\right)^2 = \frac{y_{\prime}^2}{b^2}\left(1 - \frac{x^2}{a^2}\right),$$

$$\frac{x^2}{a^2}\left(\frac{x_{\prime}^2}{a^2} + \frac{y_{\prime}^2}{b^2}\right) - 2\frac{x_{\prime}}{a^2}x + 1 - \frac{y_{\prime}^2}{b^2} = 0:$$

hence

$$SP.SQ = a^2 - ae.(x' + x'') + e^2x'x''$$

$$= a^2 - ae.\frac{2x_1}{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}} + e^2a^2.\frac{1 - \frac{y_1^2}{b^2}}{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}$$

$$= \frac{y_1^2 + (x_1 - ae)^2}{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}$$

$$= \frac{ST^2}{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}} = \frac{RN^2}{\frac{x'''^2}{a^2} + \frac{y'''^2}{b^2}} = RN^2.$$

We have replaced ST, x_1, y_1, by RN, x''', y''', where x''', y''', are the coordinates of R, by virtue of the similarity of the triangles CST, CNR.

2. A tangent at any point P of an ellipse meets the major axis produced in T, and perpendiculars upon it from the centre and focus in Y, Z: to prove that

$$\frac{TY}{PY} = \left(\frac{TZ}{PZ}\right)^2.$$

3. If P be any point in an ellipse, the major axis of which is AA', minor semi-axis BC, and foci S, S': to shew that

$$(AP + SP).(AP - SP) + (A'P + S'P).(A'P - S'P) = 2(BC)^2.$$

4. If θ be the angle between the focal distances of any point of an ellipse, and ϕ the angle between the lines joining the point with the extremities of the major axis, to prove that, e denoting the eccentricity, θ and ϕ will be connected by the equation
$$\cot\phi . (\cot\phi - e \cot\theta) = \tfrac{1}{4}e^2.$$

5. To find the locus of the centre of a circle inscribed in the triangle *SPH*, *S* and *H* being the two foci and *P* any point of a given ellipse.

The equation to the ellipse being
$$(1 - e^2)\, x^2 + y^2 = a^2 (1 - e^2),$$
the required locus will be a concentric ellipse defined by the equation
$$(1 + e)\, y^2 + (1 - e)\, x^2 = a^2e^2 (1 - e).$$

Lardner: *Algebraic Geometry*, p. 125.

6. To find the locus of the centre of a circle which, the diagram remaining the same as in the last problem, touches *SP*, *HP* produced, and *HS* produced.

The required locus is a straight line defined by the equation $x = a$, and is therefore a tangent to the ellipse through one of its vertices.

Lardner: *Algebraic Geometry*, p. 126.

7. To find the locus of the centre of a circle which, the diagram remaining the same as in the two preceding problems, touches *SH*, *PS* produced, and *PH* produced.

The locus is an ellipse defined by the equation
$$(1 - e)\, y^2 + (1 + e)\, x^2 = a^2e^2 (1 + e).$$

Lardner: *Algebraic Geometry*, p. 127.

8. If a circle be described through the two foci of an ellipse, and any point in the conjugate axis produced; to prove that the right line joining that point and one of the points where the circle cuts the ellipse, will be a tangent to the ellipse.

Leybourn: *Mathematical Repository*, *New Series*, vol. I. p. 187.

9. If SP, HP, be any two focal distances of an ellipse, the vertex of which is A, e the eccentricity, and a, b, the semi-axes, and if a straight line AQL be drawn cutting SP in Q, and bisecting HP in L: to find the locus of Q.

The required locus is an ellipse represented by the equation

$$\frac{\{x - \frac{1}{2}e(1+e)\,a\}^2}{\frac{1}{4}(1-e)^2 . a^2} + \frac{y^2}{\frac{1}{4}(1-e)^2 . b^2} = 1.$$

SECTION VII.

Referred to its Axes. Conjugate Diameters.

1. CP, CD, are conjugate semi-diameters of an ellipse; to prove that the sum of the squares of the distances of P, D, from a fixed diameter is invariable.

Let $a\cos\theta$, $b\sin\theta$, be the coordinates of P: those of D will then be $-a\sin\theta$, $b\cos\theta$. Let p, p', represent the respective distances of these points from a fixed diameter represented by the equation

$$lx + my = 0,$$

where l, m, are its direction-cosines. Then

$$p = la\cos\theta + mb\sin\theta,$$

$$p' = -la\sin\theta + mb\cos\theta,$$

and therefore

$$p^2 + p'^2 = l^2a^2 + m^2b^2.$$

2. CP and CD are conjugate semi-diameters of an ellipse. If normals at P and D intersect in K, to prove that KC is perpendicular to PD.

Let (x, y), (x_1, y_1), be the coordinates of P, D, respectively: then the equations to the normals will be

$$(x' - x)\,\frac{a^2}{x} = (y' - y)\,\frac{b^2}{y},$$

$$(x' - x_1)\,\frac{a^2}{x_1} = (y' - y_1)\,\frac{b^2}{y_1}.$$

At the intersection of these two lines

$$(a^2 - b^2)(y_1 - y) = -a^2\left(\frac{y}{x} - \frac{y_1}{x_1}\right) x',$$

$$(b^2 - a^2)(x_1 - x) = -b^2\left(\frac{x}{y} - \frac{x_1}{y_1}\right) y',$$

and therefore $$-\frac{y_1 - y}{x_1 - x} = -\frac{a^2}{b^2} \cdot \frac{yy_1}{xx_1} \cdot \frac{x'}{y'}:$$

but, by the nature of conjugate diameters,

$$x_1 = -\frac{a}{b} y, \qquad y_1 = \frac{b}{a} x:$$

hence $$\frac{y_1 - y}{x_1 - x} = -\frac{x'}{y'},$$

which shews that CK is perpendicular to PD.

3. If C be the centre of an ellipse and, in the normal to any point P, PQ be taken equal to the semi-diameter CD which is conjugate to the semi-diameter CP; to find the locus of Q.

Let (x, y) be the coordinates of P, and (x', y') those of Q. Then, by the equation to the normal, we have

$$\frac{a^2}{x}(x' - x) = \frac{b^2}{y}(y' - y).$$

This equation shews that, λ being some arbitrary quantity,

$$\left.\begin{aligned} x' - x &= \lambda \cdot \frac{bx}{a} \\ y' - y &= \lambda \cdot \frac{ay}{b} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots(1).$$

But, the coordinates of D being $-\frac{ay}{b}$, $\frac{bx}{a}$, we have, by the condition of the problem,

$$(x' - x)^2 + (y' - y)^2 = \frac{b^2x^2}{a^2} + \frac{a^2y^2}{b^2},$$

and therefore, by (1), $\lambda^2 = 1$, $\lambda = \pm 1$.

Hence, by (1), we have

$$x' = (a \pm b)\,\frac{x}{a},$$

$$y' = (b \pm a)\,\frac{y}{b}.$$

From these equations, combined with the equation to the ellipse, we have, for the equation to the required locus,

$$x'^2 + y'^2 = (a \pm b)^2.$$

This shews that the locus is either of two circles concentric with the ellipse, of which the radii are $a - b$ and $a + b$.

4. If a series of chords of an ellipse pass through a fixed point, to prove that the chords of the corresponding conjugate arcs have the same property.

Let the coordinates of the extremities of any one of the series of chords be $(a \cos\theta, b \sin\theta)$ and $(a \cos\theta', b \sin\theta')$. The equation to the chord will therefore be

$$\frac{x}{a}(\sin\theta - \sin\theta') - \frac{y}{b}(\cos\theta - \cos\theta') = \sin(\theta - \theta'),$$

and therefore, (h, k) being the fixed point through which every chord of the series passes,

$$\frac{h}{a}(\sin\theta - \sin\theta') - \frac{k}{b}(\cos\theta - \cos\theta') = \sin(\theta - \theta') \ldots (1).$$

Again, the coordinates of the extremities of the conjugate chord are $(-a \sin\theta, b \cos\theta)$, $(-a \sin\theta', b \cos\theta')$: its equation will therefore be

$$\frac{x}{a}(\cos\theta - \cos\theta') + \frac{y}{b}(\sin\theta - \sin\theta') = \sin(\theta - \theta'),$$

or, by virtue of (1),

$$\left(\frac{x}{a} + \frac{k}{b}\right)(\cos\theta - \cos\theta') + \left(\frac{y}{b} - \frac{h}{a}\right)(\sin\theta - \sin\theta') = 0.$$

This equation shews that the conjugate chord passes always through a fixed point

$$x = -\frac{ak}{b}, \qquad y = \frac{bh}{a}.$$

5. *CP* and *CD* are conjugate semi-diameters of an ellipse, and *PF* is a perpendicular let fall upon *CD*: to determine the locus of the point *F*.

Let x, y, be the coordinates of *P*. Then the equation to the normal *PF* will be

$$\frac{a^2}{x}(x'-x)=\frac{b^2}{y}(y'-y)\dots\dots\dots\dots\dots\dots(1),$$

and the equation to *CD* will be

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=0\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

At the intersection of (1) and (2), we have

$$x'(x'-x)+y'(y'-y)=0,$$

or $$x'^2+y'^2=xx'+yy',$$

and therefore, by (2),

$$\frac{x'^2+y'^2}{b^2}=\frac{xx'}{b^2}-\frac{xx'}{a^2},$$

or $$\frac{x}{a}=\frac{a(x'^2+y'^2)}{(a^2-b^2)x'}.$$

Similarly $$\frac{y}{b}=\frac{b(x'^2+y'^2)}{(b^2-a^2)y'}.$$

But the equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1:$$

hence $$\frac{(x'^2+y'^2)^2}{(a^2-b^2)^2}\cdot\left\{\frac{a^2}{x'^2}+\frac{b^2}{y'^2}\right\}=1,$$

which is the equation to the required locus.

6. The equation to an ellipse being

$$2x^2+3y^2=4,$$

to ascertain the equation to a diameter conjugate to one represented by the equation $y=2x$.

The required equation is

$$y=-\tfrac{1}{3}x.$$

7. CP, CD, are two conjugate semi-diameters of an ellipse, of which S is a focus: to prove that the distance of P from a diameter, drawn parallel to the line joining S and D, is equal to the minor semi-axis.

8. To find the value of the sum of the squares of two normals of an ellipse drawn from the extremities of two conjugate diameters to the major axis.

The required value is equal to

$$\frac{b^2}{a^2} \cdot (a^2 + b^2).$$

9. If CP, CD, be conjugate diameters of an ellipse, and ordinates through P, D, meet another ellipse, described on the same major axis, in Q, E, respectively: to prove that CQ, CE, are conjugate diameters of the second ellipse.

10. PSP', QHQ', are chords drawn through the foci S, H, of an ellipse, parallel to any pair of conjugate diameters: if $SP = r$, $SP' = r'$, $SQ = \rho$, $SQ' = \rho'$, to prove that, $2b$, $2l$, denoting the axis minor and latus-rectum respectively,

$$rr' + \rho\rho' = b^2 + l^2.$$

11. Two conjugate semi-diameters a', b', of an ellipse, are inclined at angles θ, ϕ, respectively to the semi-axis major: to prove that $$a'^2 \sin 2\theta + b'^2 \sin 2\phi = 0.$$

12. C is the centre, S the focus, and P any point of an ellipse: to prove that, Q being the intersection of PS with the diameter conjugate to CP,

$$PQ = \text{the semi-axis major.}$$

13. If (a_1, b_1) and (a_2, b_2) be the coordinates of the extremities of any two diameters of an ellipse, to find the angle between the two conjugate diameters.

The required angle is equal to

$$\tan^{-1}\left(a^2b^2 . \frac{a_1b_2 - a_2b_1}{a^4b_1b_2 + b^4a_1a_2}\right),$$

which, if the given diameters be conjugate, reduces itself to

$$\tan^{-1}\left\{\frac{a_1^2b^2 + b_1^2a^2}{(a^2 - b^2) . a_1b_1}\right\}.$$

14. If, from the extremities P, D, of two conjugate semi-diameters CP, CD, of an ellipse, straight lines be drawn, parallel to the semi-axis minor CB, to meet, in points P', D', the circumference of a circle which has the axis major for its diameter: to prove that the angle $P'CD'$ is a right angle.

15. If D represent any diameter of an ellipse and P the parameter of D, to find when $D + P$ is the least and when the greatest possible.

It is the least when D is the axis major and the greatest when it is the axis minor.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 38.

16. If CP, CD, be any conjugate semi-diameters of an ellipse $APBDA'$, A and A' being the ends of the major and B an end of the minor axis, and if BP, BD, be joined, and also AD, $A'P$, these latter two intersecting in O, to prove that $BDOP$ is a parallelogram, the greatest area of which is equal to $ab.(\sqrt{2} - 1)$.

17. If a, b, be the semi-axes of an ellipse, and a', b', conjugate semi-diameters, to prove that, a' being inclined to a at an angle α, and b' to b at an angle β,

$$\frac{a'^2 - b'^2}{a^2 - b^2} = \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)}.$$

18. To find the locus of the middle points of the chords joining the extremities of conjugate diameters in an ellipse.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the required locus will be an ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \tfrac{1}{2}.$$

19. If tangents be drawn from different points of an ellipse, and of lengths equal to n times the conjugate semi-diameters at those points; to find the locus of their extremities.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the required locus will be a concentric ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + n^2.$$

SECTION VIII.

Referred to Axes parallel to the Axes of the Curve.

1. Two ellipses lie in the same plane, are similar, and have their major axes parallel. Each is cut in four points by lines drawn from its centre to touch the other. To prove that the eight points of intersection lie on two parallel lines.

Let the equations to the two ellipses be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{..........................} \quad (1),$$

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = m^2 \quad \text{.................} (2).$$

Draw a tangent from (α, β), the centre of the latter, to touch the former.

Then, (x, y), being the point in which (1) is touched, the equation

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \text{ .... } (3)$$

will be satisfied when α and β are put for x' and y'; hence

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1 \text{ .... } (4).$$

Combining (4) with (3), we get

$$\frac{y}{b^2}(\beta x' - \alpha y') = x' - \alpha, \quad \frac{x}{a^2}(\alpha y' - \beta x') = y' - \beta,$$

and thence, by (1),

$$(\alpha y' - \beta x')^2 = a^2(y' - \beta)^2 + b^2(x' - \alpha)^2,$$

or, dropping accents,

$$(\alpha y - \beta x)^2 = a^2(y - \beta)^2 + b^2(x - \alpha)^2,$$

the equation to the system of two tangents to (1) passing through (α, β). Combining this equation with (2) multiplied by a^2b^2, we get

$$\alpha y - \beta x = \pm\, mab,$$

the equations to two parallel lines on which the four points of intersection lie. The distance of either from the centre of either of the ellipses is obviously

$$\frac{mab}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}.$$

In order to determine the corresponding system for the tangents drawn from the centre of the former to touch the latter of the two ellipses, write ma, mb, for a, b, respectively, and $\frac{1}{m}$ for m; when

$$\frac{mab}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}$$

remains unchanged. The two systems therefore coincide.

2. To find the equation to an ellipse, placed with its axes parallel to the rectangular axes of coordinates, so as to touch the lines

$$x = 2a, \quad x = 2a', \quad y = 2b, \quad y = 2b'.$$

The required equation is

$$\frac{(x-a-a')^2}{(a-a')^2}+\frac{(y-b-b')^2}{(b-b')^2}=1.$$

SECTION IX.

Polar Equation. Centre the Pole.

1. CP, CD, are two conjugate semi-diameters of an ellipse, of which CP is inclined to the major axis at an angle α: to find the magnitude of CD.

The polar equation to the ellipse, the centre being the pole, gives
$$CP^2(a^2\sin^2\alpha+b^2\cos^2\alpha)=a^2b^2.$$

Hence
$$\begin{aligned}CD^2&=a^2+b^2-CP^2\\&=a^2+b^2-\frac{a^2b^2}{a^2\sin^2\alpha+b^2\cos^2\alpha}\\&=\frac{a^4\sin^2\alpha+b^4\cos^2\alpha}{a^2\sin^2\alpha+b^2\cos^2\alpha}.\end{aligned}$$

2. If CP, CD, be two semi-diameters of an ellipse at right angles to each other, to prove that the distance of the centre from the chord PQ is equal to

$$\frac{ab}{(a^2+b^2)^{\frac{1}{2}}}.$$

Let the equation to the ellipse be

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2},$$

and that to the chord PQ

$$r\cos(\theta-\alpha)=\delta.$$

At the intersection of these two lines

$$\cos^2(\theta-\alpha)=\delta^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right).$$

Let λ, $\frac{1}{2}\pi + \lambda$, be the angular coordinates of P, D, respectively: then

$$\cos^2(\lambda - \alpha) = \delta^2 \left(\frac{\cos^2\lambda}{a^2} + \frac{\sin^2\lambda}{b^2}\right),$$

and
$$\sin^2(\lambda - \alpha) = \delta^2 \left(\frac{\sin^2\lambda}{a^2} + \frac{\cos^2\lambda}{b^2}\right).$$

Adding these two equations, we get

$$1 = \delta^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

$$\delta^2 = \frac{a^2 b^2}{a^2 + b^2}.$$

3. If A be the elliptic area contained by two semi-diameters including an angle α, and B that contained between two semi-diameters at right angles to the former, to prove that

$$\left(\frac{a}{b} + \frac{b}{a}\right) \cot \alpha = \cot \frac{2A}{ab} + \cot \frac{2B}{ab}.$$

The polar equation to the ellipse, the centre being the pole, is

$$r^2 \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1:$$

hence
$$A = \iint r\,d\theta\,dr = \tfrac{1}{2}\int r^2 d\theta = \tfrac{1}{2}\int \frac{d\theta}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}$$

$$= \tfrac{1}{2} b^2 \int_{\tan\theta'}^{\tan\theta} \frac{d\tan\theta}{\dfrac{b^2}{a^2} + \tan^2\theta},$$

where $\theta - \theta' = \alpha$: or

$$A = \tfrac{1}{2} ab \left\{\tan^{-1}\left(\frac{a\tan\theta}{b}\right) - \tan^{-1}\left(\frac{a\tan\theta'}{b}\right)\right\},$$

whence
$$\cot \frac{2A}{ab} = \frac{b}{a} \cdot \frac{1 + \dfrac{a^2}{b^2}\tan\theta\tan\theta'}{\tan\theta - \tan\theta'}.$$

Putting $\theta + \frac{1}{2}\pi$ for θ, and $\theta' + \frac{1}{2}\pi$ for θ', we have

$$\cot \frac{2B}{ab} = \frac{b}{a} \cdot \frac{\frac{a^2}{b^2} + \tan\theta \tan\theta'}{\tan\theta - \tan\theta'} :$$

hence
$$\cot \frac{2A}{ab} + \cot \frac{2B}{ab} = \frac{b}{a}\left(1 + \frac{a^2}{b^2}\right) \cot(\theta - \theta')$$
$$= \left(\frac{a}{b} + \frac{b}{a}\right) \cot\alpha.$$

4. If two semi-diameters of an ellipse be drawn at right angles to each other, to prove that, r, r', denoting their lengths,

$$\frac{1}{r^2} + \frac{1}{r'^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Abonné: *Gergonne*, *Annales de Mathématiques*, tom. XVIII. p. 369.

5. To prove that of all diameters of an ellipse, the axis major is the greatest and the axis minor the least, and that a diameter nearer to the axis major is greater than one more remote.

Apollonius: *Conicorum*, lib. V. prop. 11.

6. If CP make an angle θ with the axis major of an ellipse, C being its centre and P a point in its circumference, and be produced to meet a tangent at the extremity of the axis minor, so that the part produced is equal to the semi-axis minor; to prove that θ is always less than $\frac{\pi}{6}$; and, when $\theta = \frac{\pi}{10}$, that $e = \frac{8}{5} \sin \frac{\pi}{5}$.

7. If (θ, r), (θ', r'), be the coordinates of any two points in an ellipse, the centre being the pole and the major semi-axis the prime radius vector, to prove that, $2a$, $2b$, being the axes,

$$\frac{1}{r^2} - \frac{1}{r'^2} = (a^2 - b^2)(\sin^2\theta - \sin^2\theta').$$

8. The two axes of an ellipse are 6 and 4, and, from its centre, two straight lines, including a right angle, are drawn

to the curve. Supposing these lines to bear to each other the ratio of 3 to 4, to find their actual lengths.

The required lengths are

$$\frac{15}{2\sqrt{13}} \text{ and } \frac{10}{\sqrt{13}}.$$

9. To find the locus of the extremity of a straight line drawn from the centre of an ellipse, such that the rectangle contained by it and the diameter perpendicular to it, may be equal to half the rectangle contained by the axes.

The equation to the required locus, the centre of the ellipse being the pole, will be

$$\frac{r^2}{a^2b^2} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}.$$

SECTION X.

Polar Equation. Focus the Pole.

1. In an ellipse HP is bisected in L: AL is joined, cutting SP in Q. To find the locus of Q.

Since $SC = HC$, and $HL = PL$, it follows that CL is parallel to SP. Hence

$$\frac{SQ}{CL} = \frac{AS}{AC} = 1 - e.$$

Put $$SQ = \rho, \quad \angle ASQ = \theta:$$

then $$\rho = (1-e)\,CL = \tfrac{1}{2}(1-e).SP$$

$$= \tfrac{1}{2}(1-e).\frac{a(1-e^2)}{1+e\cos\theta}$$

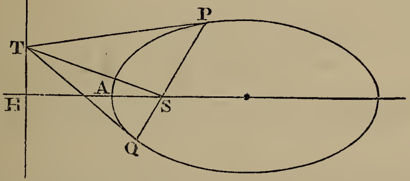
$$= \frac{\tfrac{1}{2}(1-e)^2.(1+e).a}{1+e\cos\theta};$$

which shews that the locus of Q is an ellipse, similar to the original one.

The semi-axis major $= \dfrac{\tfrac{1}{2}(1-e)^2(1+e)\,a}{1-e^2} = \tfrac{1}{2}(1-e)\,a.$

2. The eccentricity of an ellipse is $\frac{1}{\sqrt{2}}$, and, through the focus, a straight line is drawn, inclined at an angle of 45° to the axis major, meeting the ellipse in two points: to shew that the tangents at these points meet at an angle the tangent of which is 2.

The intersection of PT, QT, the two tangents, will take place in the directrix TH. Join ST, which will, by a known theorem, be at right angles to PQ.



Then

$$HS = AH + AS = AS\left(1 + \frac{1}{e}\right) = a(1-e)\,.\,\frac{1+e}{e} = \frac{a}{\sqrt{2}},$$

$$ST = \frac{HS}{\cos 45^\circ} = a,$$

$$SP = \frac{a(1-e^2)}{1 - e\cos 45^\circ} = a,$$

$$SQ = \frac{a(1-e^2)}{1 + e\cos 45^\circ} = \tfrac{1}{3}a.$$

If $\angle PTS = \alpha$, and $\angle QTS = \beta$, we shall therefore have

$$\tan\alpha = \frac{SP}{ST} = 1, \qquad \tan\beta = \frac{SQ}{ST} = \tfrac{1}{3},$$

and therefore
$$\tan(\alpha+\beta) = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2.$$

3. S and H are the two foci of an ellipse: from any point P in the curve a normal PG is drawn to cut SH in G. To prove that the projection of PG upon HP or SP is constant.

From G draw GL at right angles to PH, and let $\angle PHS = \theta$, $e =$ the eccentricity, $2a =$ the axis major, $CG = x$, C being the centre of the ellipse.

Then
$$\begin{aligned} PL &= HP - HL \\ &= HP - (ae + e^2x)\cos\theta \\ &= HP - e.HP.\cos\theta \\ &= HP(1 - e\cos\theta) \\ &= a(1 - e^2) \\ &= \text{the semi-latus-rectum.} \end{aligned}$$

The same proposition is true also in relation to the hyperbola and parabola.

Pagès: *Liouville*, *Journal de Mathématiques*, tom. II. p. 437.

4. If Pp, Qq, be chords of an ellipse drawn through one of the foci at right angles to each other, to prove that

$$\frac{1}{Pp} + \frac{1}{Qq} = \frac{2 - e^2}{2a(1 - e^2)}.$$

5. If P be any point in the periphery of an ellipse, S and H being the two foci, to prove that, the angles PSH, PHS, being denoted respectively by θ, θ',

$$\tan\frac{\theta}{2}.\tan\frac{\theta'}{2} = \frac{1 - e}{1 + e}.$$

6. If an ellipse and parabola have a common vertex and common focus, to prove that, Sp, SP, being respectively focal distances inclined at a common angle θ to the axis,

$$\frac{Pp}{Sp} = \frac{1 - e}{1 + e}.\tan^2\frac{\theta}{2}.$$

7. A chord is drawn through the focus of an ellipse making an angle θ with the major axis, and tangents are drawn at the extremities of this chord: to find the angle contained between these two tangents.

If ϕ be the required angle, then

$$\tan\phi = \frac{2e\sin\theta}{1-e^2}.$$

8. In an ellipse, of which S is a focus, SA (the least distance) $= D$, $\angle ASP = \theta$, and the eccentricity $= e$; to prove that, if $\beta = \frac{1-e}{1+e}$, and powers of β above the first be neglected,

$$SP = \frac{D}{\cos^2\frac{1}{2}\theta.(1+\beta\tan^2\frac{1}{2}\theta)}.$$

9. To find the locus of the middle points of all focal chords of an ellipse.

If C be the centre and S the focus of the original ellipse, the required locus will be a similar ellipse having SC as its major axis.

10. Five radii vectores of an ellipse being represented by a, b, c, d, e, and its parameter by p, to prove that

$$\tfrac{1}{2}p = \frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\delta + \sin\varepsilon}{\frac{\sin\alpha}{a} + \frac{\sin\beta}{b} + \frac{\sin\gamma}{c} + \frac{\sin\delta}{d} + \frac{\sin\varepsilon}{e}},$$

where

$$\alpha = \angle(b, c) + \angle(d, e),$$
$$\beta = \angle(c, d) + \angle(e, a),$$
$$\gamma = \angle(d, e) + \angle(a, b),$$
$$\delta = \angle(e, a) + \angle(b, c),$$
$$\varepsilon = \angle(a, b) + \angle(c, d).$$

This problem was proposed for solution by Gergonne, to whom it had been communicated by Mr. Talbot, in the 17th volume of the *Annales de Mathématiques*, p. 283, and solved, in a more general form, by Lenthéric, p. 366.

SECTION XI.

Polar Equation. End of the Axis Major the Pole.

1. A circle is described upon AA', the major axis of an ellipse. P is any point in this circle: AP, $A'P$, are joined, cutting the ellipse in points Q and Q' respectively: to shew that

$$\frac{AP}{AQ} + \frac{A'P}{A'Q'} = \frac{a^2 + b^2}{b^2}.$$

The polar equation to the ellipse, A being the pole, and AA' the direction of the prime radius vector, is

$$AQ = \frac{\dfrac{2\cos\theta}{a}}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}.$$

But, by the equation to the circle,

$$AP = 2a\cos\theta:$$

hence
$$\frac{AP}{AQ} = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)a^2.$$

Similarly, $\frac{1}{2}\pi - \theta$ taking the place of θ,

$$\frac{A'P}{A'Q'} = \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)a^2.$$

Hence
$$\frac{AP}{AQ} + \frac{A'P}{A'Q'} = \frac{a^2 + b^2}{b^2}.$$

2. Through A, the common vertex of two similar ellipses ABB', ADD', the greater axes of which are coincident in direction, are drawn chords ABD, $AB'D'$: B, B', and D, D', are joined: to prove that BB' is parallel to DD'.

3. If CP be any semi-diameter of an ellipse, and AQO be drawn from the extremity A of the major axis parallel to PC, and meeting the curve in Q and the minor axis produced in O: to prove that

$$AO.AQ = 2CP^2.$$

Herschel: *Leybourn's Mathematical Repository, New Series*, vol. IV. p. 152.

SECTION XII.

Polar Equation. End of the Axis Minor the Pole.

1. To determine the longest straight line which can be drawn from the extremity of the minor axis of an ellipse to the curve.

The equation to an ellipse, an end of the axis minor being the pole, and the axis minor the prime radius vector, is

$$r = \frac{2b\cos\theta}{1 - e^2 + e^2\cos^2\theta}.$$

Our object is to find when r is the greatest, or, putting

$$u = \frac{\cos\theta}{1 - e^2 + e^2\cos^2\theta},$$

when u is the greatest.

Now $$e^2u\cos^2\theta - \cos\theta = -(1 - e^2)u,$$

$$\left(eu\cos\theta - \frac{1}{2e}\right)^2 = \frac{1}{4e^2} - (1 - e^2)u^2.$$

From this it appears that u can never be greater than $\dfrac{1}{2e(1 - e^2)^{\frac{1}{2}}}$, and, since $\cos\theta$ cannot be greater than 1, can be so great only when

$$\frac{1}{2e^2u} \leqq 1,$$

or $$u \geqq \frac{1}{2e^2},$$

or $$\frac{1}{2e(1 - e^2)^{\frac{1}{2}}} \geqq \frac{1}{2e^2},$$

$$e \geqq (1 - e^2)^{\frac{1}{2}},$$

$$e^2 \geqq \tfrac{1}{2}.$$

If $e^2 < \frac{1}{2}$, the value of u will be greatest when

$$eu\cos\theta - \frac{1}{2e}$$

is least, or, since $\cos\theta$ is always smaller than $\frac{1}{2e^2u}$, when $\cos\theta$ is greatest, that is, when $\theta = 0$.

Thus, if $e^2 >$ or $= \frac{1}{2}$, we have, r being greatest,

$$\cos\theta = \frac{(1-e^2)^{\frac{1}{4}}}{e}, \quad r = \frac{b}{e(1-e^2)^{\frac{1}{4}}},$$

and, when $e^2 < \frac{1}{2}$,

$$\theta = 0, \quad r = 2b.$$

SECTION XIII.

Polar Equation. Point in the Axis the Pole.

1. O is a point, in the semi-axis major CA of an ellipse, at a distance from the vertex A equal to half the latus-rectum. P is any point in the curve. PO is joined and PM is drawn at right angles to CA. To prove that OA is less than OP, and that

$$(OP)^2 - (OA)^2 \propto (AM)^2.$$

If $OP = r$, $\angle POA = \theta$, $e =$ the eccentricity, and $CA = a$, the polar equation to the ellipse will be

$$e^2r^2\cos^2\theta - 2ae^2(1-e^2)\,r\cos\theta = r^2 - (1-e^2)(1-e^4)\,a^2.$$

This equation may be written in the following form,

$$\{er\cos\theta - ae(1-e^2)\}^2 = r^2 - (1-e^2)^2.a^2;$$

and therefore, observing that

$$r\cos\theta = OM, \quad a(1-e^2) = OA,$$

we see that
$$e^2.(AM)^2 = (OP)^2 - (OA)^2.$$

This result shews that OP is greater than OA.

Apollonius: *Conicorum*, lib. v. prop. 6.

2. In the semi-axis major CA of an ellipse, a point O is taken, at a distance from the vertex A greater than the ratio of $(BC)^2$ to AC. O is joined to any point P in the curve, and PM is drawn perpendicularly to CA. To determine the least value of OP, and the corresponding value of OM.

Let $CO = c$; then, when PO is the least possible,

$$PO = \frac{(1-e^2)^{\frac{1}{2}}}{e} . (e^2a^2 - c^2)^{\frac{1}{2}},$$

$$MO = c . \frac{1-e^2}{e^2}.$$

Apollonius: *Conicorum*, lib. v. prop. 10.

3. In the semi-axis minor BCB' of an ellipse, a point O is taken at a distance from B equal to the ratio of $(AC)^2$ to BC; to prove that OB is the greatest distance of the point O from the curve.

Apollonius: *Conicorum*, lib. v. prop. 16, 17, 18.

4. A point O is taken in the axis minor BCB' of an ellipse, at a distance from B greater than the semi-axis minor BC, but less than the ratio of $(AC)^2$ to BC. O is joined to any point P in the curve: to determine the magnitude and position of the greatest value of OP.

Let $\angle POB = \theta$, and $OP = r$, $CO = c$. Then the required magnitude and position are given by the equations

$$r = \left(a^2 + \frac{c^2}{e^2}\right)^{\frac{1}{2}}, \qquad \cos\theta = \frac{c}{e\,(a^2e^2 + c^2)^{\frac{1}{2}}}.$$

Apollonius: *Conicorum*, lib. v. prop. 20.

SECTION XIV.

Polar Equation. Pole anywhere.

1. To prove that two chords of an ellipse, which are not diameters, cannot bisect each other in their point of intersection.

Taking the point of intersection of the chords as the pole, we may write the equation to the ellipse in the form

$$\frac{(\rho\cos\theta + h)^2}{a^2} + \frac{(\rho\sin\theta + k)^2}{b^2} = 1.$$

Suppose that, disregarding signs, the two values of ρ given by this equation are equal, for a certain value of θ; then, the signs of the equal values of ρ being opposite, we have

$$\frac{h \cos \theta}{a^2} + \frac{k \sin \theta}{b^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Suppose the same thing to be true for some other chord through the pole, ϕ replacing θ: then

$$\frac{h \cos \phi}{a^2} + \frac{k \sin \phi}{b^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$\sin \phi \cos \theta = \sin \theta \cos \phi,$$

$$\sin (\phi - \theta) = 0,$$

$$\phi = \theta.$$

This result shews that the latter chord must coincide with the former, and therefore establishes the proposition.

Ἐὰν ἐν ᾿ελλείψει ἢ κύκλου περιφερείᾳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὖσαι· οὐ τέμνουσιν ἀλλήλας δίχα.

ʼΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ *Κωνικῶν τὸ δεύτερον. Πρότασις δ′.*

2. Through a given point O, is drawn a straight line cutting an ellipse in two points P, Q: to determine the position of the line in order that the rectangle $OP.OQ$ may be a maximum or minimum.

Let θ = the inclination of the line to the axis major: then the rectangle will be a maximum if $\theta = 0$, and a minimum if $\theta = \frac{1}{2}\pi$.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 31.

3. If there be two similar concentric ellipses, similarly placed, the ratio between either axis of the one to the corresponding axis in the other being $\sqrt{2}$ to 1; to prove that, if from any point in the periphery of the inner ellipse, any right line be drawn to terminate in the periphery of the outer ellipse, the

rectangle under the segments of that line, is equal to the square of that semi-diameter of the inner ellipse, which is parallel to the right line.

Ivory: Leybourn's *Mathematical Repository*, *New Series*, vol. II. p. 169.

SECTION XV.

Referred to Conjugate Diameters.

1. *CP*, *CD*, are two conjugate semi-diameters of an ellipse. If *PD* be joined, and, through any point *Q* of the curve, be drawn a line parallel to *PD* cutting *CP*, *CD*, respectively in *P'*, *D'*, to prove that

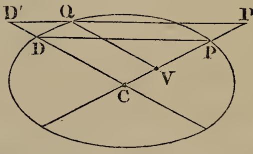
$$(QP')^2 + (QD')^2 = (PD)^2.$$

Draw *QV* parallel to *DC*, to cut *CP* in *V*.

Let $CP = a$, $CD = b$, and accordingly, λ being some arbitrary quantity,

$$CP' = \lambda a, \quad CD' = \lambda b.$$

Also let $CV = x$, $QV = y$, $PD = c$, $QP' = r$.



Then, the triangles *QVP'*, *DCP*, being similar,

$$x = \lambda a - \frac{a}{c} . r, \quad y = \frac{b}{c} r:$$

but, by the equation to the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1:$$

hence $$\frac{1}{a^2}\left(\lambda a - \frac{a}{c} r\right)^2 + \frac{r^2}{c^2} = 1,$$

and therefore $$r^2 - \lambda c r + \tfrac{1}{2}(\lambda^2 - 1)\, c^2 = 0.$$

The two roots $r_{,}$, $r_{,,}$, of this equation, which is symmetrically related to the axes of x and y, represent the lines *QP'*, *QD'*: but

$$r_{,}^2 + r_{,,}^2 = (r_{,} + r_{,,})^2 - 2r_{,}r_{,,} = \lambda^2 c^2 - (\lambda^2 - 1)\, c^2 = c^2:$$

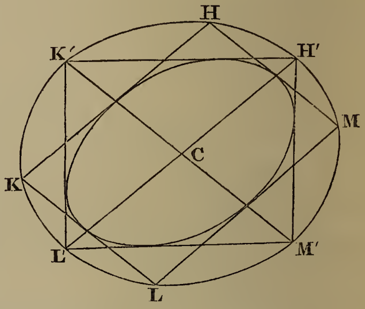
the proposition is therefore established.

In the case of an hyperbola, we might prove in a like way that

$$r_{\prime}^2 - r_{\prime\prime}^2 = c^2.$$

Encontre: *Gergonne, Annales de Mathématiques*, tom. IV. p. 294.

2. If, in a given ellipse, be inscribed any parallelogram *HKLM*, the sides of which are parallel to conjugate diameters; and if an ellipse be inscribed in this parallelogram, so as to touch its sides in their middle points; to prove that this new ellipse will also touch the sides of a parallelogram $H'K'L'M'$, the diagonals $H'L'$, $K'M'$, of which are conjugate diameters of the given ellipse parallel to the sides *HK*, *KL*, of the parallelogram *HKLM*.



Let

$$HK = 2a, \qquad HM = 2b,$$
$$H'L' = 2a', \quad K'M' = 2b'.$$

Then, since the ellipse inscribed within *HKLM* touches the sides in their middle points, it follows that the lines joining their middle points are conjugate diameters: hence the equation to the inscribed ellipse, CH', CK', being taken as axes of coordinates, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ .... } (1).$$

Again, the equation to $H'K'$ is

$$\frac{x}{a'} + \frac{y}{b'} = 1 \text{ .... } (2).$$

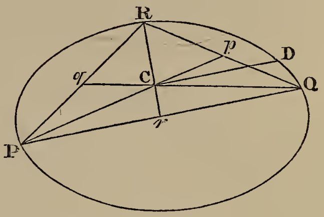
But, since H, the coordinates of which are a, b, lies in the given ellipse,

$$\frac{a^2}{a'^2} + \frac{b^2}{b'^2} = 1 \text{ .... } (3).$$

The relation (3) expresses the condition of tangency between the lines (1) and (2). Similarly the other sides of $H'K'L'M'$ may be shewn to be tangents to the inscribed ellipse.

Rochat: *Gergonne, Annales de Mathématiques*, tom. III. p. 27.

3. To find the area of the greatest triangle which can be inscribed in a given ellipse.



Let PQR be the greatest triangle which can be inscribed in the ellipse. Then Q must be the point in which a tangent parallel to PR touches the ellipse, for Q will in this case be the point of greatest perpendicular distance from PR. Similar remarks hold good in relation to the points P, R, and the sides QR, PQ, respectively opposite to them.

Let C be the centre of the ellipse: join PC, QC, RC, and produce these lines to meet the sides of the triangle in p, q, r.

Then, from what has been said, it is clear that RQ is a conjugate chord to the diameter PCp, PR to QCq, and PQ to RCr. Hence p, q, r, are the middle points of RQ, RP, PQ; and therefore, by a known property of triangles,

$$Cr = \tfrac{1}{2}CR, \quad Cq = \tfrac{1}{2}CQ, \quad Cp = \tfrac{1}{2}CP.$$

Hence, by the equation to the ellipse referred CR, and its conjugate CD,

$$\begin{aligned}\text{area of } \Delta PQR &= Rr.Pr.\sin\angle PrR \\ &= \tfrac{3}{2}CR\,.\,\frac{CD}{CR}\,.\,(CR^2 - \tfrac{1}{4}CR^2)^{\frac{1}{2}}.\sin\angle DCR \\ &= \frac{3\sqrt{3}}{4}\,.\,CR\,.\,CD\,.\sin\angle DCR \\ &= \tfrac{3}{4}\sqrt{3}.ab.\end{aligned}$$

COR. The area of the ellipse is equal to πab: hence

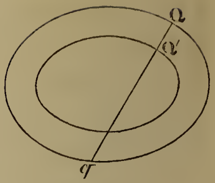
area of greatest triangle : area of ellipse :: $3\sqrt{3} : 4\pi$.

A different solution of this problem will be given subsequently in this volume.

4. If lines, drawn through any point in an ellipse to the extremities of any diameter PCP', meet the direction of its conjugate diameter DCD' in points M, N, to prove that

$$CM\,.\,CN = CD^2.$$

5. If a series of parallel chords be drawn cutting two similar and concentric ellipses; to prove that the product of the part QQ', intercepted between the two curves, and the remaining part $Q'q$, will be constant.



6. If a parallelogram be described about an ellipse, its sides being parallel to a pair of conjugate diameters, to prove that the portions of its diagonals, which are intercepted by the ellipse, form another pair of conjugate diameters.

7. From the ends of two conjugate diameters of an ellipse are drawn lines parallel to any tangent line: and, from the centre C, is drawn any line cutting these lines and the tangent in points p, q, r, respectively: to prove that

$$(Cp)^2 + (Cq)^2 = (Cr)^2.$$

8. CP, CD, being any two conjugate semi-diameters of an ellipse, PD is joined, a semi-diameter CP' is drawn parallel to DP, and PP' is joined: to find the area of the trapezium $CP'PD$.

Required area $= (\sqrt{2} + 1) \cdot \dfrac{ab}{2\sqrt{2}}$.

9. If two points be taken on any semi-diameter of an ellipse, so that the rectangle of the segments between them and the centre may be equal to the square of the semi-diameter; and from these points straight lines be drawn to any point in the periphery of the ellipse, meeting it again in two other points: to prove that the straight line joining these two other points will be parallel to the tangent at the vertex of the diameter.

Leybourn: *Mathematical Repository*, *New Series*, vol. III. p. 31.

10. To prove that the tangent to the interior of two similar ellipses, the axes of which are coincident, cuts off from the exterior curve a constant area.

De la Hire: *Sectiones Conicæ*, lib. VI. prop. 17.

11. A tangent to an ellipse, at a point P', intersects conjugate semi-diameters CP, CD, produced, in H, K, respectively: CD' is conjugate to CP'. To prove that

$$(CD')^2 = HP' . KP'.$$

De la Hire: *Sectiones Conicæ*, lib. V. prop. 17.

SECTION XVI.

Referred to any two Diameters.

1. If, at the extremity of each two diameters (not conjugate) of an ellipse, a tangent be drawn meeting the other diameter, the line joining the points of intersection will be parallel to that joining the extremities of the diameters.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + 2\frac{x}{\alpha}\frac{y}{\beta} + \frac{y^2}{b^2} = 1,$$

a, b, being the semi-diameters taken as coordinate axes.

The equation to the tangent at any point (x, y) will be

$$(x' - x)\left(\frac{x}{a^2} + \frac{y}{\alpha\beta}\right) + (y' - y)\left(\frac{x}{\alpha\beta} + \frac{y}{b^2}\right) = 0,$$

or
$$x'\left(\frac{x}{a^2} + \frac{y}{\alpha\beta}\right) + y'\left(\frac{x}{\alpha\beta} + \frac{y}{b^2}\right) = 1.$$

Put $x = 0$, $y = b$, $y' = 0$: then, x' denoting the distance of the intersection of the axis of x from the origin,

$$\frac{bx'}{\alpha\beta} = 1, \qquad x' = \frac{\alpha\beta}{b}.$$

Similarly, y' denoting a like quantity in relation to the axis of y,

$$y' = \frac{\alpha\beta}{a}.$$

Hence
$$\frac{x'}{a} = \frac{y'}{b},$$

an equation which establishes our proposition.

2. To find the magnitude and position of the principal diameters of an ellipse defined by the equation

$$ax^2 + by^2 + 2cxy = f \text{ .... } (1),$$

the axes of coordinates being supposed to include an angle α.

Let r be the distance of any point of the ellipse from its centre. Then

$$r^2 = x^2 + y^2 + 2xy \cos\alpha \text{.................. } (2).$$

At the extremities of the principal diameters, r is a maximum or minimum: hence, differentiating the two equations, and putting $dr = 0$, we obtain

$$0 = xdx + ydy + (xdy + ydx) \,.\, \cos\alpha,$$

$$0 = axdx + bydy + (xdy + ydx) \,.\, c.$$

From these equations, λ being an arbitrary multiplier, we get

$$\lambda (x + y \cos\alpha) = ax + cy,$$

and

$$\lambda (y + x \cos\alpha) = by + cx.$$

Multiplying the former of these two equations by x, and the latter by y, and adding, we see that, by virtue of (1) and (2),

$$\lambda r^2 = f.$$

Hence

$$x (f - ar^2) = y (cr^2 - f \cos\alpha),$$

and

$$y (f - br^2) = x (cr^2 - f \cos\alpha).$$

Eliminating x and y, we get

$$(f - ar^2)(f - br^2) = (cr^2 - f \cos\alpha)^2.$$

From this equation we shall have two values r'^2, r''^2, of r^2. The quantities $2r'$, $2r''$, will be the lengths of the principal diameters. The equations to these diameters will be

$$x (f - ar'^2) = y (cr'^2 - f \cos\alpha),$$

$$x (f - ar''^2) = y (cr''^2 - f \cos\alpha).$$

Berard: *Gergonne, Annales de Mathématiques*, tom. III. p. 105.
Bret: *Gergonne, Annales de Mathématiques*, tom. V. p. 357.

SECTION XVII.

Referred to any Rectangular Axes whatever. Reduction.

The equation to an ellipse referred to any axes whatever is

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

a, b, c, being subject to the condition

$$c^2 \angle ab.$$

If we substitute $x' + \alpha$, $y' + \beta$, for x, y, respectively, and, in the resulting equation, put equal to zero the coefficients of x', y', we shall reduce the equation to the form

$$a_1x'^2 + b_1y'^2 + 2c_1x'y' + c_1' = 0,$$

the corresponding values of α, β, being the coordinates of the centre of the ellipse.

Again, putting, in the last equation, the axes being supposed to be rectangular,

$$x' = x'' \cos\theta - y'' \sin\theta,$$

$$y' = x'' \sin\theta + y'' \cos\theta,$$

and then equating to zero the coefficient of $x''y''$ in the resulting equation, we shall reduce it to the simple form

$$a_2x''^2 + b_2y''^2 + c_2' = 0,$$

the corresponding values of θ determining the positions of the principal axes.

1. To construct the ellipse of which the equation is

$$5x^2 + 2xy + 5y^2 - 12x - 12y = 0.$$

Transform the origin to a point α, β: the equation will then become

$$5(x'+\alpha)^2+2(x'+\alpha)(y'+\beta)+5(y'+\beta)^2-12(x'+\alpha)-12(y'+\beta)=0.$$

Equating to zero the coefficients of x' and y', we have

$$10\alpha + 2\beta - 12 = 0, \quad \text{or} \quad 5\alpha + \beta - 6 = 0,$$

and $$2\alpha + 10\beta - 12 = 0, \quad \text{or} \quad \alpha + 5\beta - 6 = 0.$$

From these two equations we see that

$$\alpha = 1, \quad \beta = 1.$$

The equation to the curve then becomes

$$5x'^2 + 2x'y' + 5y'^2 - 12 = 0.$$

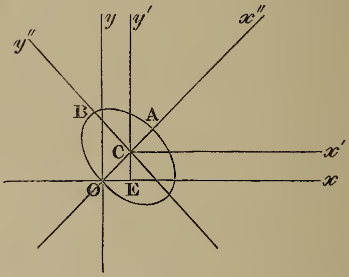
Again, turning the axes through an angle θ, we shall have for the transformed equation

$$5(x''\cos\theta - y''\sin\theta)^2 + 2(x''\cos\theta - y''\sin\theta).(x''\sin\theta + y''\cos\theta) + 5(x''\sin\theta + y''\cos\theta)^2 - 12 = 0.$$

Equating to zero the coefficient of $x''y''$, we obtain

$$-10\sin\theta\cos\theta + 2(\cos^2\theta - \sin^2\theta) + 10\sin\theta\cos\theta = 0,$$

whence $\cos 2\theta = 0, \quad 2\theta = \frac{1}{2}\pi, \quad \theta = \frac{1}{4}\pi.$



The equation to the curve then becomes

$$6x''^2 + 4y''^2 = 12,$$

or

$$\frac{x''^2}{2} + \frac{y''^2}{3} = 1.$$

The coordinates of the centre of the ellipse, referred to the original axes, are therefore 1, 1; and the semi-axes are $\sqrt{2}$, $\sqrt{3}$, the former being inclined to the axis of x at an angle $\frac{1}{4}\pi$.

The diagram will serve to illustrate the geometrical signification of the transformations

$$AC = \sqrt{2}, \quad BC = \sqrt{3}, \quad OE = 1 = CE, \quad \angle xOx'' = \frac{1}{4}\pi.$$

2. To find the positions and magnitudes of the principal axes of the ellipse denoted by the equation

$$7x^2 + 5y^2 + 2\sqrt{3}.xy + 6x + 10\sqrt{3}.y + 7 = 0.$$

The coordinates of its centre are $(0, -\sqrt{3})$; and its semi-axes are 4, 1, the latter of which is inclined to the axis of x at an angle $\frac{1}{6}\pi$.

3. To find the positions and magnitudes of the principal axes of the ellipse represented by the equation

$$3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$$

The coordinates of its centre are $(-1, 3)$; and its semi-axes are $\frac{1}{2}$, $\frac{1}{\sqrt{2}}$, the former of which is inclined to the axis of x at an angle $\frac{1}{4}\pi$.

4. To find what the equation

$$9x^2 + 3y^2 - 6xy - 6x - 2y + \tfrac{7}{3} = 0$$

becomes, when the curve is referred to its principal axes.

The required equation is

$$x^2\left(1 + \frac{1}{\sqrt{2}}\right) + y^2\left(1 - \frac{1}{\sqrt{2}}\right) = \tfrac{1}{9}.$$

5. The equation to an ellipse being

$$x^2 + y^2 + xy + x + y = 1,$$

to find the coordinates of its centre, and its equation referred to its axes.

The coordinates of its centre are $-\frac{1}{3}, -\frac{1}{3}$, and the required equation is

$$3x^2 + y^2 = \tfrac{8}{3}.$$

Garnier: *Géométrie Analytique*, p. 117.

6. To find the equation to the ellipse

$$2x^2 + y^2 - 2xy - 2x = 0$$

when referred to its axes.

The required equation is

$$(3 - \sqrt{5})\,x^2 + (3 + \sqrt{5})\,y^2 = 2.$$

Garnier: *Géométrie Analytique*, p. 129.

SECTION XVIII.

Linear Equation.

1. If r, ρ, be the distances of the two foci of an ellipse from a point P in the curve; and r', ρ', from a point D in the curve; P and D being the extremities of conjugate diameters; to prove that

$$r\rho + r'\rho' = a^2 + b^2.$$

Let $a\cos\theta$ be the abscissa of P: then we know that $-a\sin\theta$ will be that of D: hence

$$r = a(1 - e\cos\theta), \quad \rho = a(1 + e\cos\theta),$$

whence

$$r\rho = a^2(1 - e^2\cos^2\theta).$$

Similarly, $-\sin\theta$ now occupying the place of $\cos\theta$,

$$r'\rho' = a^2(1 - e^2\sin^2\theta).$$

Hence

$$r\rho + r'\rho' = a^2 + a^2(1 - e^2)$$
$$= a^2 + b^2.$$

Brassine: *Liouville, Journal de Mathématiques*, Avril, 1842.

2. If in an ellipse there be taken three abscissæ in arithmetical progression along the major axis, the radii vectores, drawn from one focus to the extremities of the corresponding ordinates, will also be in arithmetical progression.

3. To prove that, P and D being the extremities of any two conjugate diameters in an ellipse,

$$(SP - AC)^2 + (SD - AC)^2 = SC^2.$$

SECTION XIX.

Intersections of Ellipses.

1. If two ellipses, so placed in a plane that two conjugate diameters of the one are parallel to two conjugate diameters of the other, intersect each other in four points; these four points will lie in a third ellipse, the equal conjugate diameters of which

shall be parallel respectively to the said conjugate diameters in the two former ellipses.

Let the axes of coordinates be taken parallel to the parallel conjugate diameters of the two proposed ellipses. Their equations will be

$$ax^2 + by^2 + 2a'x + 2b'y + c = 0,$$

$$\alpha x^2 + \beta y^2 + 2\alpha' x + 2\beta' y + \gamma = 0.$$

Multiplying these two equations by r and ρ, and adding, we get

$$(ra+\rho\alpha)x^2+(rb+\rho\beta)\,y^2+2\,(ra'+\rho\alpha')\,x+2\,(rb'+\rho\beta')\,y+rc+\rho\gamma=0,$$

the equation to an ellipse in which the two proposed ellipses intersect.

Putting $$ra + \rho\alpha = rb + \rho\beta,$$

we have $$\frac{r}{\rho} = \frac{\beta - \alpha}{a - b}:$$

if therefore this value be given to the ratio $\frac{r}{\rho}$, the coefficients of x^2 and y^2 in the equation to this third ellipse will be equal, and consequently its two equal conjugate diameters will be parallel to the coordinate axes.

Par un Abonné: *Gergonne*, *Annales de Mathématiques*, tom. v. p. 88.

2. Two ellipses, of equal eccentricity and of which the major axes are parallel, can have only two points in common. To prove this, and to shew that, if three such ellipses intersect two and two in the points A and A', B and B', C and C', respectively, the lines AA', BB', CC', intersect in one point.

The equations to two such ellipses, referred to axes of coordinates parallel to the principal axes, will be

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{(x-\alpha')^2}{a'^2} + \frac{(y-\beta')^2}{b'^2} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and, e denoting the eccentricity of either,

$$1-e^2=\frac{b^2}{a^2}=\frac{b'^2}{a'^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (1), (2), (3), we have

$$(1-e^2)(x-\alpha)^2+(y-\beta)^2=b^2\ \ldots\ldots\ldots\ldots\ (4),$$

$$(1-e^2)(x-\alpha')^2+(y-\beta')^2=b'^2\ldots\ldots\ldots\ldots\ (5).$$

Subtracting (5) from (4),

$$(1-e^2)\{2(\alpha'-\alpha)x-(\alpha'^2-\alpha^2)\}+2(\beta'-\beta)y-(\beta'^2-\beta^2)=b^2-b'^2\ldots(6),$$

from which equation it appears that all points of intersection of (4) and (5) lie in a straight line; and, since a straight line can cut an ellipse in only two points, the former part of the proposition is established.

Let the equation to the third ellipse be

$$(1-e^2)(x-\alpha'')^2+(y-\beta'')^2=b''^2\ \ldots\ldots\ldots\ (7).$$

The equations to the lines AA', BB', CC', will be (6), (8), (9),

$$(1-e^2)\{2(\alpha''-\alpha')x-(\alpha''^2-\alpha'^2)\}$$
$$+2(\beta''-\beta')y-(\beta''^2-\beta'^2)=b'^2-b''^2\ldots(8),$$

$$(1-e^2)\{2(\alpha-\alpha'')x-(\alpha^2-\alpha''^2)\}$$
$$+2(\beta-\beta'')y-(\beta^2-\beta''^2)=b''^2-b^2\ \ldots\ (9).$$

Adding together the equations (6), (8), (9), we obtain an identical equation. This shews that the three lines AA', BB', CC', intersect each other in a single point.

3. Two equal similar ellipses, which have the same centre, have their axes inclined at a given angle: to find the angle between the curves where they intersect.

Let C be the common centre of the two ellipses, E their point of intersection, ET the tangent to one of the ellipses at E, meeting its major axis, produced, in T. Let CE be produced indefinitely to N.

Let $\alpha=$ the inclination of the axes of the ellipses; then it is evident that $\angle ECT=\frac{1}{2}\alpha$ or $\frac{1}{2}(\pi-\alpha)$. Let $\angle ETC=\theta$. Then the angle between the curves at E, being evidently double the angle NET, is equal to

$$2\left(\tfrac{1}{2}\alpha+\theta\right)=\alpha+2\tan^{-1}\left(\frac{1-e^2}{\tan\frac{1}{2}\alpha}\right),$$

or

$$2\left(\frac{\pi-\alpha}{2}+\theta\right)=\pi-\alpha+2\tan^{-1}\left(\frac{1-e^2}{\cot\frac{1}{2}\alpha}\right).$$

4. To find the coordinates of the points of intersection of the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\qquad \frac{x^2}{b^2}+\frac{y^2}{a^2}=1,$$

and to determine the tangent of their inclination to each other at these points.

If θ be the required angle, and x, y, the coordinates of the required points,

$$x=\pm\frac{ab}{(a^2+b^2)^{\frac{1}{2}}}=y,$$

and

$$\tan\theta=\frac{a^4-b^4}{2a^2b^2}.$$

5. There are any number of ellipses having a common centre and their major axes in the same position. To prove that, if all the ellipses be twisted about their common centre through the same angle θ in the same direction, the locus of the intersection of each ellipse with its original position, is two straight lines defined by the equations

$$y=x\tan\tfrac{1}{2}\theta,\qquad y=-x\cot\tfrac{1}{2}\theta.$$

6. If a quadrilateral be inscribed in two ellipses, which cut each other in four points, and of which the major axes are parallel or perpendicular to one another: to prove that any two of its opposite angles are together equal to two right angles.

7. Two concentric ellipses, which have their axes in the same directions, intersect, and four common tangents are drawn so as to form a rhombus, and the points of intersection of the ellipses are joined so as to form a rectangle: to prove that the product of the areas of the rhombus and rectangle is equal to half the continued product of the four axes.

SECTION XX.

Polar Equation to the Tangent.

1. If two straight lines be drawn at right angles to one another through the focus of an ellipse, and tangents to the ellipse be drawn at their extremities; to find the equation to the locus of the intersection of these tangents.

The equation to one of the tangents being

$$\frac{a(1-e^2)}{r} = \cos(\alpha - \theta) + e\cos\theta,$$

or $$\cos(\alpha - \theta) = \frac{a(1-e^2)}{r} - e\cos\theta \;\ldots\ldots\ldots\ldots\; (1)^*;$$

that to the corresponding tangent will be, $\frac{1}{2}\pi + \alpha$ being written for α,

$$-\sin(\alpha - \theta) = \frac{a(1-e^2)}{r} - e\cos\theta \ldots\ldots\ldots\ldots (2).$$

Squaring and then adding (1) and (2), we get, for the equation to the required locus,

$$1 = 2\left\{\frac{a(1-e^2)}{r}\right\}^2 + 2e^2\cos^2\theta - 4e\left\{\frac{a(1-e^2)}{r}\right\}\cos\theta,$$

$$\tfrac{1}{2} = \left\{\frac{a(1-e^2)}{r} - e\cos\theta\right\}^2,$$

$$\frac{a(1-e^2)}{r} = \pm\frac{1}{\sqrt{2}} + e\cos\theta,$$

$$\frac{a\sqrt{2}(1-e^2)}{r} = \pm 1 + e\sqrt{2}\cos\theta,$$

the equation to an ellipse, parabola, or hyperbola, as e^2 is less than, equal to, or greater than $\frac{1}{2}$.

2. If tangents be drawn, from any point without an ellipse, to the curve: to prove that their lengths are inversely as the

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[*] This equation to the tangent of an ellipse is given by Mr. Davies, in the *Philosophical Magazine* for 1842, p. 192.

sines of the angles which they make with the lines drawn from their points of contact to either focus.

Let T be the point of concourse of the two tangents, P, P', their points of contact. Then, S being the focus, and CSA the semi-axis major, let

$$PT = \rho, \quad P'T = \rho', \quad ST = r,$$

$$\angle AST = \theta, \quad \angle PSA = \alpha, \quad \angle P'SA = \alpha'.$$

Then, the polar equations to the tangents give us, at the common point T, the relation

$$e \cos\theta + \cos(\theta - \alpha) = \frac{a(1-e^2)}{r} = e \cos\theta + \cos(\theta - \alpha'),$$

whence $\theta - \alpha = -\theta + \alpha', \quad \theta = \frac{1}{2}(\alpha + \alpha'),$

and therefore $\angle PST = \frac{1}{2}(\alpha - \alpha') = \angle P'ST.$

Hence
$$\frac{\rho}{r} = \frac{\sin\dfrac{\alpha - \alpha'}{2}}{\sin\Psi}.$$

$$\frac{\rho'}{r} = \frac{\sin\dfrac{\alpha - \alpha'}{2}}{\sin\Psi'}.$$

and therefore
$$\frac{\rho}{\rho'} = \frac{\sin\Psi'}{\sin\Psi}.$$

3. To find the locus of the point E, in which the radius vector SP of an ellipse cuts the diameter conjugate to the semi-diameter CP.

The equation to the ellipse being

$$r = \frac{a(1-e^2)}{1 + e\cos\theta},$$

the equations to SP and to the tangent at P will be respectively of the forms

$$\theta = \alpha \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{a(1-e^2)}{r} = \cos(\theta - \alpha) + e\cos\theta \ldots\ldots\ldots\ldots (2).$$

The equation to the diameter conjugate to CP, which passes through the point (ae, π), and is parallel to (2), is

$$-(e+\cos\alpha)\,.\,\frac{ae}{r}=\cos(\theta-\alpha)+e\cos\theta\ldots\ldots\ldots(3).$$

At the intersection of (1) and (3), there is

$$-(e+\cos\theta)\,.\,\frac{ae}{r}=1+e\cos\theta,$$

or

$$r+a=\frac{a(1-e^2)}{1+e\cos\theta},$$

the equation to the required locus.

4. The tangent, at any point P of an ellipse, cuts the major axis, produced, in T: S is the focus. To prove that

$$\cos SPT=e\cos STP.$$

5. From a point O, two tangents OP, OP', are drawn to an ellipse, P being nearer to an end of the axis major than P' is: to prove that OP' is greater than OP.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 44.

6. To find the locus of the intersection of two tangents at points P, P', of an ellipse, such that, S being one of the foci, the sum of the inclinations of SP, SP', to the major axis may be invariable.

Take SA as the prime radius vector, A being the extremity of the major axis nearest to S; then, putting

$$\angle ASP+\angle ASP'=2\beta,$$

the locus of the intersection will be a straight line defined by the equation

$$\theta=\beta.$$

SECTION XXI.

Polar Equation to a Chord.

1. PSp is any focal chord of an ellipse, A the extremity of the axis major; AP, Ap, meet the directrix in two points Q, q: to prove that $\angle QSq$ is a right angle.

If $\alpha-\beta$, $\alpha+\beta$, be the angular coordinates of the extremities of any chord, the focus being the pole and the prime radius vector coinciding with the major axis, the equation to the chord will be

$$\frac{c}{r} = \sec\beta \,.\, \cos(\theta-\alpha) + e\cos\theta,$$

c denoting $a(1-e^2)$.

Let 2γ be the angular coordinate of P, then, 0 being that of A, we have $\alpha-\beta=0, \quad \alpha+\beta=2\gamma,$

and therefore $\alpha=\gamma=\beta.$

Hence the equation to AP is

$$\frac{c}{r} = \sec\gamma \,.\, \cos(\theta-\gamma) + e\cos\theta \ldots\ldots\ldots\ldots\ldots(1).$$

Again, the angular coordinates of A, p, being

$$\alpha-\beta=0, \quad \alpha+\beta=-(\pi-2\gamma),$$

we have $\alpha=-(\tfrac{1}{2}\pi-\gamma)=\beta.$

Hence the equation to Ap is

$$\frac{c}{r} = -\operatorname{cosec}\gamma \,.\, \sin(\theta-\gamma) + e\cos\theta \ldots\ldots\ldots\ldots(2).$$

Again, the equation to the directrix is

$$r\cos\theta = \frac{c}{e} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

At Q, the intersection of (1) and (3), we have

$$e\cos\theta = \sec\gamma\cos(\theta-\gamma) + e\cos\theta,$$

and therefore $\cos(\theta-\gamma)=0,$

$$\theta-\gamma=-\tfrac{1}{2}\pi, \quad \theta=-(\tfrac{1}{2}\pi-\gamma),$$

or $\angle ASQ = \tfrac{1}{2}\pi-\gamma.$

Again, at q, the intersection of (2) and (3),

$$0 = \sin(\theta - \gamma),$$

$$\theta - \gamma = 0, \quad \theta = \gamma,$$

or

$$\angle ASq = \gamma.$$

Hence

$$\angle QSq = \angle ASQ + \angle ASq,$$

$$= \tfrac{1}{2}\pi.$$

2. PSQ, $P'SQ'$, are any two focal chords of an ellipse: to find the locus of the intersection of the chords PP', QQ'.

The required locus is the directrix nearest to S.

SECTION XXII.

Polar Equation to the Normal.

1. To deduce the polar equation to the normal of an ellipse from that to the tangent, the focus being the pole.

Let the polar equation to the normal be

$$r(A\cos\theta + B\sin\theta) + C = 0.$$

Let α be the angular coordinate of the point of contact of the corresponding tangent. Then, c denoting $a(1-e^2)$,

$$\frac{c}{1+e\cos\alpha}(A\cos\alpha + B\sin\alpha) + C = 0,$$

and therefore

$$A\left(r\cos\theta - \frac{c\cos\alpha}{1+e\cos\alpha}\right) + B\left(r\sin\theta - \frac{c\sin\alpha}{1+e\cos\alpha}\right) = 0 \ldots\ldots(1).$$

Since this line is perpendicular to the tangent, which is defined by the equation

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha)$$

$$= (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta,$$

we must have the relation

$$\frac{A}{B} + \frac{\sin\alpha}{e+\cos\alpha} = 0,$$

and therefore (1) becomes

$$\sin\alpha.\left(r\cos\theta - \frac{c\cos\alpha}{1+e\cos\alpha}\right) = (e+\cos\alpha).\left(r\sin\theta - \frac{c\sin\alpha}{1+e\cos\alpha}\right),$$

or $$r\{(e+\cos\alpha)\sin\theta - \sin\alpha\cos\theta\} = \frac{ec\sin\alpha}{1+e\cos\alpha},$$

or $$\frac{c}{r} = \frac{1+e\cos\alpha}{e\sin\alpha}.\{e\sin\theta + \sin(\theta-\alpha)\},$$

which is the required equation to the normal.

2. If the normal through any point P in an ellipse cuts the major axis in G, to prove that, S being the focus,

$$SG = e.SP.$$

SECTION XXIII.

Poles and Polars.

1. There are two ellipses the axes of which are in the same straight lines. To find the locus of a pole of the exterior ellipse the corresponding polar of which is always a tangent to the interior.

Let a, b, be the semi-axes of the interior, and α, β, of the exterior ellipse. Let the coordinates of the pole be $x_{\prime}$, $y_{\prime}$, and let x', y', be those of the point in which its polar touches the interior ellipse.

Then, the axes of the ellipses being taken as axes of coordinates, the equation to the polar will be

$$\frac{xx_{\prime}}{\alpha^2} + \frac{yy_{\prime}}{\beta^2} = 1 \text{.......................}(1):$$

but, since it is a tangent to the interior ellipse, its equation will also be

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \text{.......................}(2).$$

Since the equations (1) and (2) must be identical, we have

$$x' = \frac{a^2}{\alpha^2}x_{\prime}, \text{ and } y' = \frac{b^2}{\beta^2}y_{\prime}:$$

hence, by the equation $\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} = 1,$

we have
$$\frac{a^2x_{\prime}^{\,2}}{\alpha^4}+\frac{b^2y_{\prime}^{\,2}}{\beta^4}=1,$$
for the equation to the locus of the pole; which is therefore a similarly situated ellipse with semi-axes $\frac{\alpha^2}{a}$, $\frac{\beta^2}{b}$.

2. To find the locus of a pole in relation to a given ellipse, the part of the polar which is intercepted by the ellipse being of a constant length.

If the equation to the ellipse be
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$
and $2c$ be the length of the intercepted portion of the polar, the equation to the required locus will be
$$c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}\right).\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

3. The polar corresponding to a pole P, relatively to a given ellipse, intersects the ellipse in points E and F: to find the locus of P, supposing the area of the triangle EPF to be constant.

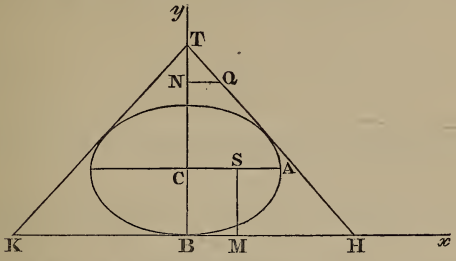
The equation to the ellipse being
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$
and m^2 denoting the constant area, the locus of P will be an ellipse defined by the equation
$$\frac{m^4}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^3.$$

SECTION XXIV.

Inscribed and Circumscribed Polygons.

1. To find the locus of the foci of all ellipses, described with their major axes parallel to the base of an isosceles triangle, and touching its three sides.

Let *HKT* be the isosceles triangle. Let *AC*, *BC*, be semi-axes of one of the inscribed ellipses, *S* a focus. Draw *SM* at right angles to *BH*. From any point *Q*, in *TH*, draw *QN* at right angles to *TB*.



Let $\angle HTB = \tan^{-1}\alpha = \angle KTB$; $BM = x$, $SM = y$, $CN = x'$, $QN = y'$, $CA = a$, $CB = b$, $BT = c$.

Then, *TH* being a tangent to the ellipse,

$$y' + \alpha x' = (\alpha^2 b^2 + a^2)^{\frac{1}{2}}.$$

When $x' = -b$, we see that $y' = \alpha c$: hence

$$\alpha(c - b) = (\alpha^2 b^2 + a^2)^{\frac{1}{2}}.$$

But $$x = (a^2 - b^2)^{\frac{1}{2}}, \quad y = b;$$

hence $$a^2 = x^2 + b^2 = x^2 + y^2;$$

and therefore, for the equation to the required locus, we get

$$\alpha(c - y) = (\alpha^2 y^2 + x^2 + y^2)^{\frac{1}{2}},$$

or $$\alpha^2 c^2 - 2\alpha^2 cy = x^2 + y^2,$$

which is the equation to a circle.

2. To determine the ellipse of greatest area which can be inscribed in a given triangle.

Let two of the sides of the triangle be taken as coordinate axes. Then the equations to the three sides will be of the forms

$$x = 0, \quad y = 0, \quad ax + by = 1;$$

and the equation to the ellipse will be

$$(\alpha x)^{\frac{1}{2}} + (\beta y)^{\frac{1}{2}} + (ax + by - 1)^{\frac{1}{2}} = 0,$$

where α and β are arbitrary constants.

Clearing the equation to the ellipse of radical indices we shall get

$$(\alpha - a)^2 x^2 + (\beta - b)^2 y^2 + 2\{(\alpha - a)(\beta - b) - 2\alpha\beta\} xy$$
$$+ 2(\alpha - a) x + 2(\beta - b) y + 1 = 0.$$

The coordinates of the centre of this ellipse will be

$$h = \frac{\frac{1}{2}(\beta - b)}{a\beta + b\alpha - ab}, \qquad k = \frac{\frac{1}{2}(\alpha - a)}{a\beta + b\alpha - ab},$$

and the equation to the ellipse, referred to the centre as origin, will be

$$(\alpha - a)^2 x'^2 + (\beta - b)^2 y'^2 + 2\{(\alpha - a)(\beta - b) - 2\alpha\beta\} x'y' = \frac{\alpha\beta}{a\beta + b\alpha - ab}.$$

Let ω be the angle between the coordinate axes: then, A denoting the area of the ellipse, we have, the axes being determined as in prob. 2, Section XVI.,

$$A^2 = \tfrac{1}{4}\pi^2 \frac{\alpha\beta \sin^2 \omega}{(ab - a\beta - b\alpha)}.$$

In order that A may be a maximum, we must have

$$3 \log (ab - a\beta - b\alpha) - \log \alpha - \log \beta = \text{minimum}:$$

hence, differentiating with regard to α, we get

$$\frac{3b}{a\beta + b\alpha - ab} - \frac{1}{\alpha} = 0,$$

$$2b\alpha - a\beta + ab = 0.$$

Similarly, by differentiating with regard to β,

$$2a\beta - b\alpha + ab = 0.$$

From these two equations we have

$$\alpha = -a, \qquad \beta = -b.$$

Hence the equation to the ellipse becomes

$$a^2x^2 + b^2y^2 + abxy - ax - by + \tfrac{1}{4} = 0,$$

or

$$(ax - \tfrac{1}{2})^2 + (by - \tfrac{1}{2})^2 + abxy = \tfrac{1}{4},$$

or

$$(ax - \tfrac{1}{3})^2 + (by - \tfrac{1}{3})^2 + (ax - \tfrac{1}{3})(by - \tfrac{1}{3}) = \tfrac{1}{12}.$$

Also $$A^2 = \tfrac{1}{4}\pi^2 . \frac{ab \sin^2 \omega}{27\, a^3 b^3} = \frac{\pi^2 \sin^2 \omega}{108\, a^2 b^2},$$

$$A = \frac{\pi \sin \omega}{6\sqrt{3}.ab}.$$

Supposing f and g to represent the intercepts of the line

$$ax + by = 1,$$

on the coordinate axes, the equation to the ellipse is

$$\left(\frac{x}{f} - \tfrac{1}{2}\right)^2 + \left(\frac{y}{g} - \tfrac{1}{2}\right)^2 + \frac{x}{f}\frac{y}{g} = \tfrac{1}{4},$$

or $$\left(\frac{x}{f} - \tfrac{1}{3}\right)^2 + \left(\frac{y}{g} - \tfrac{1}{3}\right)^2 + \left(\frac{x}{f} - \tfrac{1}{3}\right)\left(\frac{y}{g} - \tfrac{1}{3}\right) = \tfrac{1}{12},$$

and $$A = \frac{\pi f g \sin \omega}{6\sqrt{3}}.$$

Cor. 1. From the two forms of the equations to the ellipse, it is evident that it touches the sides of the triangle at their middle points, and that its centre is coincident with the centre of gravity of the triangle.

Cor. 2. Let B denote the area of the triangle. Then

$$B = \tfrac{1}{2} f g \sin \omega.$$

Hence $$A : B :: \pi : 3\sqrt{3}.$$

Bérard: *Gergonne, Annales de Mathématiques*, tom. IV. p. 284.

3. Among all quadrilaterals inscribed in an ellipse, to determine that which contains the greatest area.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let the angular points of the quadrilateral be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Then, u denoting the area of the quadrilateral,

$$2u = x_2 y_1 - x_1 y_2 + x_3 y_2 - x_2 y_3 + x_4 y_3 - x_3 y_4 + x_1 y_4 - x_4 y_1,$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1, \quad \frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} = 1, \quad \frac{x_4^2}{a^2} + \frac{y_4^2}{b^2} = 1.$$

In order that u may be a maximum, we have, differentiating these last four equations, putting $du = 0$, and using the method of indeterminate multipliers,

$$\frac{\lambda_1 x_1}{a^2} + y_4 - y_2 = 0, \qquad \frac{\lambda_1 y_1}{b^2} - x_4 + x_2 = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Hence we have

$$\frac{x_1}{a^2}(x_4 - x_2) = -\frac{y_1}{b^2}(y_4 - y_2) \quad \dots\dots\dots\dots (1),$$

$$\frac{x_2}{a^2}(x_1 - x_3) = -\frac{y_2}{b^2}(y_1 - y_3) \quad \dots\dots\dots\dots (2),$$

$$\frac{x_3}{a^2}(x_2 - x_4) = -\frac{y_3}{b^2}(y_2 - y_4) \quad \dots\dots\dots\dots (3),$$

$$\frac{x_4}{a^2}(x_3 - x_1) = -\frac{y_4}{b^2}(y_3 - y_1) \quad \dots\dots\dots\dots (4).$$

From (1) and (3), we have

$$\frac{y_3}{x_3} = \frac{y_1}{x_1} \quad \dots\dots\dots\dots\dots\dots (5),$$

and from (2) and (4),

$$\frac{y_4}{x_4} = \frac{y_2}{x_2} \quad \dots\dots\dots\dots\dots\dots (6).$$

Also, from (1) and (2),

$$\frac{1}{a^2}(x_1 x_4 - x_2 x_3) = -\frac{1}{b^2}(y_1 y_4 - y_2 y_3).$$

From the last three equations we see that

$$\frac{1}{a^2}(x_1 x_4 - x_2 x_3) = -\frac{1}{b^2} y_1 y_2 \left(\frac{x_4}{x_2} - \frac{x_3}{x_1}\right),$$

and therefore

$$\frac{y_1 y_2}{x_1 x_2} = -\frac{b^2}{a^2} \dots\dots\dots\dots\dots\dots (7).$$

The equations (5) and (6) shew that the diagonals of the quadrilateral are diameters of the ellipse, and (7) shews that they are conjugate diameters.

Durrande: *Gergonne, Annales de Mathématiques*, tom. XII. p. 223.

4. If any hexagon be circumscribed about an ellipse, the three diagonals, joining opposite angles, will all pass through one point.

The equation to the tangent of an ellipse is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \text{..........................} (1),$$

x' and y' being connected by the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \text{..........................} (2).$$

Equation (2) is equivalent to the following system of equations

$$\frac{x'}{a} + \frac{y'}{b}\sqrt{-1} = \alpha_1,$$

$$\frac{x'}{a} - \frac{y'}{b}\sqrt{-1} = \frac{1}{\alpha_1},$$

α_1 being an arbitrary quantity: hence

$$x' = \tfrac{1}{2}a\left(\alpha_1 + \frac{1}{\alpha_1}\right), \quad y' = \frac{b}{2\sqrt{-1}}\left(\alpha_1 - \frac{1}{\alpha_1}\right):$$

hence, the equation (1) becomes

$$\frac{x}{2a}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \frac{y}{2b\sqrt{-1}}\left(\alpha_1 - \frac{1}{\alpha_1}\right) = 1.$$

Put $a = a'$, $b\sqrt{-1} = b'$; and the equation to the tangent, which we will suppose to be one of the sides of the hexagon, assumes the form

$$\frac{x}{2a'}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \frac{y}{2b'}\left(\alpha_1 - \frac{1}{\alpha_1}\right) = 1 \text{............} (3).$$

The equation to the next side of the hexagon will be, in like manner, of the form

$$\frac{x}{2a'}\left(\alpha_2 + \frac{1}{\alpha_2}\right) + \frac{y}{2b'}\left(\alpha_2 - \frac{1}{\alpha_2}\right) = 1 \text{............} (4).$$

At the intersection of (3) and (4), $x_{1,2}$, $y_{1,2}$, being the coordinates of the corresponding angle of the hexagon, we shall get, by the combination of the equations,

$$x_{1,2} = a'\,\frac{1+\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \qquad y_{1,2} = b'\,\frac{1-\alpha_1\alpha_2}{\alpha_1+\alpha_2}.$$

The equation to the diagonal, through the angular points $(x_{1,2},\ y_{1,2})$, $(x_{4,5},\ y_{4,5})$, will be

$$x\,(y_{4,5}-y_{1,2}) - y\,(x_{4,5}-x_{1,2}) = x_{1,2}\,y_{4,5} - x_{4,5}\,y_{1,2};$$

or, substituting for the coordinates of the angular points their values

$$\begin{aligned} b'x\,\{(\alpha_1-\alpha_4)(1+\alpha_2\alpha_5)+(\alpha_2-\alpha_5)(1+\alpha_1\alpha_4)\} - a'y\,\{(\alpha_1-\alpha_4)(1-\alpha_2\alpha_5) \\ +\,(\alpha_2-\alpha_5)(1-\alpha_1\alpha_4)\} = 2a'b'\,(\alpha_1\alpha_2-\alpha_4\alpha_5)\ldots\ldots(5). \end{aligned}$$

By similarity, it is evident that the equation to the diagonal, through $(x_{2,3},\ y_{2,3})$, $(x_{5,6},\ y_{5,6})$, will be

$$\begin{aligned} b'x\,\{(\alpha_2-\alpha_5)(1+\alpha_3\alpha_6)+(\alpha_3-\alpha_6)(1+\alpha_2\alpha_5)\} - a'y\,\{(\alpha_2-\alpha_5)(1-\alpha_3\alpha_6) \\ +\,(\alpha_3-\alpha_6)(1-\alpha_2\alpha_5)\} = 2a'b'\,(\alpha_2\alpha_3-\alpha_5\alpha_6)\ldots\ldots(6). \end{aligned}$$

At the intersection of these two diagonals, multiplying (5) by $\alpha_3-\alpha_6$, (6) by $\alpha_4-\alpha_1$, adding the latter of the resulting equations to the former, and dividing the final equation by $\alpha_2-\alpha_5$, we shall thus get

$$\begin{aligned} b'x\,\{(\alpha_3-\alpha_6)(1+\alpha_4\alpha_1)+(\alpha_4-\alpha_1)(1+\alpha_3\alpha_6)\} - a'y\,\{(\alpha_3-\alpha_6)(1-\alpha_4\alpha_1) \\ +\,(\alpha_4-\alpha_1)(1-\alpha_3\alpha_6)\} = 2a'b'\,(\alpha_3\alpha_4-\alpha_6\alpha_1)\ldots\ldots(7). \end{aligned}$$

But, by symmetry, we know that (7) is the equation to the third diagonal: hence the diagonals (5), (6), intersect in (7).

Cambridge Mathematical Journal, vol. IV. p. 165.

5. To find the axes of the least ellipse which can be circumscribed about a given rectangle.

If $2f$, $2g$, be the sides of the rectangle, and $2a$, $2b$, the axes of the ellipse,

$$a = f\sqrt{2}, \qquad b = g\sqrt{2}.$$

Euler: *Acta Academiæ Petropolitanæ, pro anno* 1780, *pars posterior*, p. 3.

6. To find the ratio of the side of an equilateral hexagon inscribed in an ellipse, with two sides parallel to the major axis, to the side of an equilateral hexagon inscribed in a circle of which the axis major is a diameter.

If e be the eccentricity of the ellipse, the required ratio is equal to

$$\frac{4-2e^2}{4-e^2}.$$

7. To determine the ellipse of least area which can be circumscribed about a given triangle.

If two of the sides be taken as axes of coordinates, and f, g, be the intercepts of these axes made by the third side, the equation to the ellipse will be

$$\left(\frac{x}{f}-\tfrac{1}{3}\right)^2+\left(\frac{y}{g}-\tfrac{1}{3}\right)^2+\left(\frac{x}{f}-\tfrac{1}{3}\right)\left(\frac{y}{g}-\tfrac{1}{3}\right)=\tfrac{1}{3}.$$

The centre of this ellipse coincides with the centre of gravity of the area of the triangle, and its tangents at the three angles are respectively parallel to the opposite sides. Moreover the area of the ellipse is to that of the triangle as $4\pi : 3\sqrt{3}$.

Euler: *Nova Acta Petropolitana*, tom. IX. p. 146.
Gergonne: *Annales de Mathématiques*, tom. IV. p. 288.
Gregory: *Examples in the Differential and Integral Calculus*, p. 123, 2^d *edition.*

8. If one triangle be inscribed within another, so that their sides are parallel each to each, a single ellipse will be, at the same time, the greatest which can be inscribed in the greater triangle and the least which can be circumscribed about the smaller triangle.

Gergonne: *Annales de Mathématiques*, tom. IV. p. 291.

9. To prove that, in every parallelogram circumscribed about an ellipse, the diagonals intersect each other in the centre of the ellipse and are in the directions of two conjugate diameters.

Ferriot: *Gergonne, Annales de Mathématiques*, tom. XVI. p. 373.

10. To find the ellipse of least area which can be circumscribed about a given quadrilateral.

Let the diagonals be taken as axes of coordinates: then $(\alpha, 0)$, $(-\alpha', 0)$, $(0, \beta)$, $(0, -\beta')$, being the four angular points of the quadrilateral, the equation to the ellipse will be

$$\frac{x^2}{\alpha\alpha'} + 2cxy + \frac{y^2}{\beta\beta'} + \frac{x}{\alpha\alpha'}(\alpha' - \alpha) + \frac{y}{\beta\beta'}(\beta' - \beta) = 1.$$

The equation to the ellipse will therefore be of the form

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

where c is the only unknown parameter.

The determination of c will depend upon the solution of the cubic equation

$$c'c^3 - 4a'b'c^2 + (3a'^2b + 3ab'^2 - abc')\,c - 2aba'b' = 0.$$

Euler: *Nova Acta Academiæ Petropolitanæ*, tom. IX. p. 132.

11. The straight lines which bisect the angles of a triangle, meet the opposite sides in the points P, Q, R: to prove that an ellipse may be inscribed in the triangle such as to touch the sides in P, Q, R.

SECTION XXV.

Elliptic Loci.

1. From P, a point in an ellipse, lines are drawn to A, A', the extremities of the major axis, and from A, A', lines are drawn perpendicular to AP, $A'P$: to find the locus of the intersection of these lines.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and $x_{,}$, $y_{,}$, being the coordinates of P, the equation to AP will be

$$y = \frac{y_{,}}{x_{,} - a}(x - a),$$

and the equation to the perpendicular through A will accordingly be

$$y = \frac{a - x_{\prime}}{y_{\prime}} (x - a) \dots\dots\dots\dots\dots\dots\dots (1).$$

Similarly, $-a$ replacing a, the equation to the other perpendicular will be

$$y = -\frac{a + x_{\prime}}{y_{\prime}} (x + a) \dots\dots\dots\dots\dots\dots (2).$$

At the intersection of (1) and (2),

$$y^2 = -\frac{a^2 - x_{\prime}^2}{y_{\prime}^2} (x^2 - a^2)$$

$$= -\frac{a^2}{b^2} (x^2 - a^2),$$

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2}{b^2},$$

the equation to the required locus, which is therefore a concentric and similarly placed ellipse of which the semi-axes are a, $\frac{a^2}{b}$.

2. If all the ordinates of a circle, referred originally to rectangular coordinates, be moved through a given angle, the abscissa and magnitude of each ordinate remaining the same; to find the locus of the extremities of the ordinates, the origin of coordinates being the centre.

If c denote the radius of the circle, the equation to the required locus, referred to the original axis of x and to an axis of y, parallel to the new ordinates, is evidently

$$x^2 + y^2 = c^2,$$

which represents an ellipse referred to its two equal conjugate diameters as axes.

Let a, b, denote the semi-axes of the ellipse, and θ, θ', the inclinations of the axes of x, y, respectively, to the direction of a.

Then, ω denoting the angle between the two new axes,

$$ab = c^2 \sin \omega,$$

$$a^2 + b^2 = 2c^2,$$

which equations are obviously satisfied by

$$a = c\sqrt{2}\cos\tfrac{1}{2}\omega, \quad b = c\sqrt{2}\sin\tfrac{1}{2}\omega.$$

Also $$\tan\theta.\tan(\omega+\theta) = -\frac{b^2}{a^2} = -\tan^2\tfrac{1}{2}\omega,$$

and therefore $$\theta = -\tfrac{1}{2}\omega, \quad \theta' = \omega + \theta = \tfrac{1}{2}\omega.$$

We have thus ascertained the magnitudes and positions of the axes of the ellipse.

3. A straight line of given length $2c$ is made to move so that its ends lie always in two other given straight lines including an angle α: to find the locus of its middle point.

Let the two given lines be taken as axes of coordinates. Let the intercepts of the moveable line on these axes, in any one of its positions, be a, b. Then

$$4c^2 = a^2 + b^2 - 2ab\cos 2\alpha.$$

But, if x, y, be the coordinates of the middle point of the moveable line,

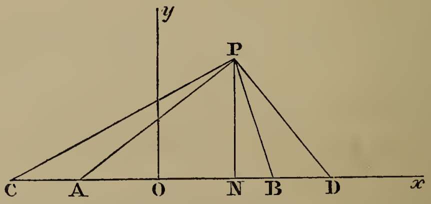
$$x = \tfrac{1}{2}a, \quad y = \tfrac{1}{2}b:$$

hence $$c^2 = x^2 + y^2 - 2xy\cos 2\alpha,$$

which is the equation to the locus required. The locus is therefore an ellipse.

Cor. It may be shewn that the semi-axes of the ellipse are $c\cot\alpha$, $c\tan\alpha$, the former one bisecting the angle between the positive directions of the coordinate axes.

4. A, B, are two given points: to find the locus of the point P which is so situated that perpendiculars (drawn through it) to AP, BP, always cut off a given length on AB.



Take O, the middle point of AB, as origin, OBx as the axis of x, and Oy, perpendicular to it, as that of y.

Let $$AC = u, \quad BD = v,$$
$$AB = 2a, \quad CD = 2b.$$

Then, from the right-angled triangle CPB, we see that
$$y^2 = (u + a + x).(a - x),$$
or
$$\frac{y^2}{a - x} = u + a + x \text{..................... } (1);$$
and, from the right-angled triangle APD, we see that
$$y^2 = (v + a - x).(a + x),$$
or
$$\frac{y^2}{a + x} = v + a - x \text{..................... } (2).$$

From (1) and (2), we have
$$\frac{y^2}{a - x} + \frac{y^2}{a + x} = u + v + 2a:$$
but
$$u + v + 2a = 2b:$$
hence
$$\frac{y^2}{a - x} + \frac{y^2}{a + x} = 2b,$$
or
$$ay^2 = b(a^2 - x^2),$$
the equation to an ellipse the axes of which coincide with the axes of coordinates.

5. The focus of a parabola lies always in the circumference of a circle represented by the equation
$$x^2 + y^2 = c^2,$$
the axis of the parabola being parallel to the axis of x, and its arc passing through the centre of the circle. To find the locus of the vertex of the parabola.

The equation to the parabola will be of the form
$$\mu x + 2\nu y + y^2 = 0.$$

Change the origin to a point h, k: its equation will then become $\mu(x'+h)+2\nu(y'+k)+(y'+k)^2=0.$

Assume the coefficient of y', and the terms which involve neither x' nor y', to be zero: then

$$\mu h+2\nu k+k^2=0,$$

$$\nu+k=0,$$

and therefore $$\mu=\frac{k^2}{h}.$$

The equation to the curve becomes

$$y'^2=-\frac{k^2}{h}x'.$$

Let $x_{\prime}$, $y_{\prime}$, be the coordinates of the focus: then, since h, k, are the coordinates of the vertex,

$$x_{\prime}=h-\frac{k^2}{4h}, \quad y_{\prime}=k;$$

and therefore, the focus being a point in the circle,

$$c^2=x_{\prime}^2+y_{\prime}^2=\left(h-\frac{k^2}{4h}\right)^2+k^2$$

$$=h^2+\tfrac{1}{2}k^2+\frac{k^4}{16h^2},$$

$$\pm c=h+\frac{k^2}{4h}$$

$$h^2+\frac{k^2}{4}=\pm ch,$$

the equation to the locus of the vertex, which accordingly lies always in the arc of one of two ellipses.

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 20.

6. A square $CBPQ$ moves in the right angle xOy, so that the point C is always in Oy, and the point B always in Ox: to find the locus of P.

If a represent the length of a side of the square, then, Ox, Oy, being taken as coordinate axes, the locus of P will be an ellipse of which the equation is

$$x^2-2xy+2y^2=a^2.$$

7. To determine the locus of the extremity of the straight line formed by augmenting the ordinate of the circle

$$x^2 + y^2 = c^2,$$

by a length equal to the abscissa.

The locus will be an ellipse defined by the equation

$$2x^2 - 2xy + y^2 = c^2.$$

Fuss: *Nova Acta Academiæ Petropolitanæ*, tom. XV. p. 72.

8. To find the locus of the intersection of two tangents to a circle, such that the product of the portions (measured from the centre) cut off by them from a given diameter, is always constant.

If the given diameter be taken as the axis y; then, a denoting the radius of the circle and k^2 a constant quantity, the locus will be an ellipse of which the equation is

$$(a^2 + k^2)\, x^2 + a^2y^2 = k^2a^2.$$

9. In the given right lines AP, AQ, are taken variable points p, q, such that

$$Ap : pP :: Qq : qA;$$

to find the locus of the point of intersection of Pq, Qp.

Let APx, AQy, be taken as axes of x and y: let $AP = a$, $AQ = b$. Then the required locus will be an ellipse represented by the equation

$$\frac{b}{a}\, x\, (a - x) + \frac{a}{b}\, y\, (b - y) = (a - x)\, (b - y).$$

Cor. If we put $x = 0$, $y = 0$, successively, we get respectively $(y - b)^2 = 0$, $(x - a)^2 = 0$. Hence the ellipse touches Ax, Ay, at the points P and Q.

10. A straight line AB of given length, moves with its extremities A, B, always in contact with two straight lines Ox, Oy, respectively, inclined to each other at a given angle: to find the locus of an assigned point P in AB.

Let $PA = a$, $PB = b$, and $\angle xOy = \omega$: then, Ox, Oy, being taken as coordinate axes, the required locus will be an ellipse defined by the equation

$$\frac{a^2x^2 + b^2y^2}{a^2b^2} - 2\frac{xy}{ab}\cos\omega = 1.$$

Newton: *Arithmetica Universalis*, prob. 21.
Bret: *Gergonne, Annales de Mathématiques*, tom. VI. p. 15.

11. To find the locus of the intersection of the lines represented by the equations

$$x\sin\theta - y\cos\theta = (x^2 + y^2)^{\frac{1}{2}},$$

and

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{x^2 + y^2}.$$

The required locus is an ellipse the equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

12. From a point (a, b) a normal is drawn to a parabola

$$y^2 = lx;$$

to find the locus of the intersection of the parabola and normal, l being a variable parameter.

The required locus is an ellipse defined by the equation

$$y(y - b) + 2x(x - a) = 0.$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 259.

13. To find the locus of the middle point of a chord of an ellipse, the chord passing through a given point.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the coordinates of the given point being (α, β), the locus required will be another ellipse defined by the equation

$$\frac{(2x - \alpha)^2}{a^2} + \frac{(2y - \beta)^2}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}.$$

14. A triangle ABC moves in such a manner that the angular points A, B, are always in two rectangular axes Ox, Oy, respectively: to find the locus of C.

Let $BC = a$, $CA = b$, $AB = c$, and $\angle ACB = \omega$. Then the equation to the required locus will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{x}{a} \cdot \frac{y}{b} \cdot \sin\omega = (\cos\omega)^2.$$

15. From the focus S of a given ellipse is drawn SQ always bisecting the angle PSC, C being the centre, and equal to the mean proportional between SC and SP: to find the locus of Q.

The equation to the given ellipse being

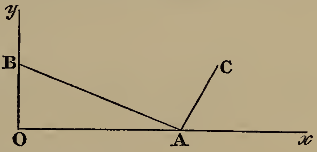
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the required locus will be another ellipse defined by the equation

$$\frac{(x - ae)^2}{1 + e} + \frac{y^2}{1 - e} = a^2 e.$$

16. A right angle BAC, the sides AB, AC, of which are given, moves in such a manner that A, B, lie always in two perpendicular lines Ox, Oy, respectively: to find the locus of C.

Take Ox, Oy, as axes of x, y, respectively. Let $AB = a$, $AC = b$. Then the locus of C will be an ellipse defined by the equation



$$b^2x^2 + (a^2 + b^2)\, y^2 - 2abxy = b^4.$$

Newton: *Arithmetica Universalis*, prob. 22.

17. The circumstances remaining the same as in the preceding problem, except that $\angle AOB$, and $\angle BAC$, instead of being right angles, are each of them equal to α; to find the locus of C.

The locus is an ellipse defined by the equation

$$b^2x^2 + (a^2 + b^2 - 2ab\cos\alpha)\, y^2 + 2b\,(b\cos\alpha - a)\, xy = b^4.$$

Newton: *Arithmetica Universalis*, prob. 22.

18. The base and the sum of two sides of a triangle being constant, to find the locus of the centre of the escribed circle touching the base.

Let the middle point of the base be taken as the origin of rectangular coordinates, the axis of y being perpendicular to the base. Let $2a$ represent the length of the base, $2b$ the sum of the two sides. Then the required locus will be an ellipse defined by the equation

$$y^2 = \frac{b+a}{b-a} \cdot (a^2 - x^2).$$

19. One extremity A of a straight line ABP and a given point B in it, move in straight lines intersecting at right angles in C: to prove that the locus of any other given point P in the line is an ellipse, and that, if the rectangle $ACBD$ be completed, and PD joined, PD is a normal at P.

20. To find the locus of the points of quadrisection of a system of parallel chords in a circle.

The equation to the circle being

$$x^2 + y^2 = a^2,$$

and that to any one of the parallel chords

$$lx + my = \delta,$$

where $l = \cos\alpha$, $y = \sin\alpha$, the equation to the required locus will be

$$(3m^2 + 1)\,x^2 - 6lmxy + (3l^2 + 1)\,y^2 = c^2.$$

21. Having given the point where a parabola intersects a given diameter, and also the parameter of that diameter, to find the locus of the vertex of the curve.

Let p represent the given parameter; then, the given diameter and a perpendicular to it through its vertex being chosen as axes of x and y respectively, the required locus will be an ellipse denoted by the equation

$$4x^2 + y^2 + px = 0.$$

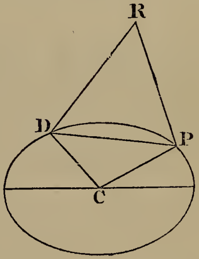
Lardner: *Algebraic Geometry*, p. 131.

22. Two semicircles are described on the segments of the base of a semicircle of which the radius is r: to find the locus of the centre of a circle touching these three semicircles.

If the middle point of the base of the large semicircle be taken as the origin of rectangular coordinates, the base being the axis of x, the locus required will be the arc of an ellipse of which the equation is

$$\frac{x^2}{\frac{4}{3}r^2} + \frac{(y + \frac{2}{3}r)^2}{\frac{16}{9}r^2} = 1.$$

23. *CP*, *CD*, are conjugate semi-diameters of an ellipse: on the chord *PD* an equilateral triangle *PDR* is described: to find the equation to the locus of *R*.



The equation to the proposed ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the required locus will also be an ellipse of which the equation is

$$\frac{x^2}{(a + b\sqrt{3})^2} + \frac{y^2}{(b + a\sqrt{3})^2} = \tfrac{1}{2}.$$

SECTION XXVI.

Elliptic Envelops.

1. *CP*, *CD*, are any two conjugate semi-diameters in an ellipse, of which the semi-axes are a and b: to find the locus of the consecutive intersections of all such chords as *PD*.

If x', y', be the coordinates of P, we may put

$$x' = a\cos\theta, \quad y' = b\sin\theta:$$

we shall then have also, x'', y'', being the coordinates of D,

$$x'' = -\frac{a}{b}y' = -a\sin\theta, \quad y'' = \frac{b}{a}x' = b\cos\theta.$$

Hence the equation to PD is, by the formula

$$(x'' - x')\, y - (y'' - y')\, x = x''y' - x'y'',$$

$$y\,(-a \sin\theta - a \cos\theta) - x\,(b \cos\theta - b \sin\theta)$$
$$= -a \sin\theta \,.\, b \sin\theta - a \cos\theta \,.\, b \cos\theta,$$

or $$(ay - bx) \sin\theta + (ay + bx) \cos\theta = ab \;\ldots\ldots\ldots(1).$$

Differentiating with respect to θ, we have

$$(ay - bx) \cos\theta - (ay + bx) \sin\theta = 0 \;\ldots\ldots\ldots(2).$$

Squaring and adding (1) and (2), we get, for the equation to the required locus,

$$2a^2y^2 + 2b^2x^2 = a^2b^2.$$

2. If, from every point in the axis major of an ellipse, as centre, a circle be described, with radius equal to the ordinate at that point: to find the envelop of the circles.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

that to the envelop will be an arc of another ellipse represented by the equation

$$\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1.$$

3. Two straight lines are drawn from one extremity of the major axis of an ellipse, making with it angles the tangents of which are in a given ratio: to find the locus of the ultimate intersections of chords, each of which joins the two points where two corresponding lines cut the ellipse.

The equation to the ellipse being

$$y^2 = \frac{b^2}{a^2}\,(2ax - x^2),$$

the envelop required will be another ellipse, the equation of which referred to the same axes will be, α denoting the given ratio,

$$y^2 = \frac{4\alpha b^2}{(1 + \alpha)^2 . \, a^2}\,(2ax - x^2).$$

4. AEC, BFC, EF, are three straight lines given in position: from the given points A, B, straight lines are drawn, through any point in EF, meeting BC in Q, AC in P: PQ is joined. To prove that the curve, to which PQ is always a tangent, is an ellipse inscribed in the quadrilateral $APQB$, and touching the lines AC, BC, at the points E, F.

Leybourn: *Mathematical Repository*, *New Series*, vol. III. p. 143.

SECTION XXVII.

Miscellaneous Problems.

1. Of all systems of conjugate diameters in an ellipse, the principal diameters are those of which the sum is a minimum; and the equal conjugate diameters are those of which the sum is a maximum.

Let a, b, be the principal semi-diameters of an ellipse, and a', b', any two conjugate semi-diameters, including an angle γ. Then we know that

$$a'^2 + b'^2 = a^2 + b^2,$$

and

$$a'b' \sin\gamma = ab.$$

From these two equations we have

$$a' + b' = \left(a^2 + b^2 + \frac{2ab}{\sin\gamma}\right)^{\frac{1}{2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$a' - b' = \left(a^2 + b^2 - \frac{2ab}{\sin\gamma}\right)^{\frac{1}{2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation (2) shews that $\sin\gamma$ cannot be less than

$$\frac{2ab}{a^2 + b^2}.$$

Thus $\sin\gamma$ is necessarily included between the limits

$$1 \text{ and } \frac{2ab}{a^2 + b^2}.$$

Now $a' + b'$ is greatest when $\sin\gamma$ is least: in this case, from (1) and (2),

$$a' + b' = \{2(a^2 + b^2)\}^{\frac{1}{2}},$$

and $$a' - b' = 0;$$

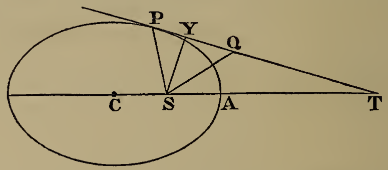
whence $$a' = b' = \left(\frac{a^2 + b^2}{2}\right)^{\frac{1}{2}}.$$

Also $a' + b'$ is least when $\sin\gamma$ is greatest: in this case, from (1) and (2),
$$a' + b' = a + b,$$
and $$a' - b' = a - b,$$
whence $$a' = a, \quad b' = b.$$

Durrande: *Gergonne, Annales de Mathématiques*, tom. XII. p. 168.

2. If a straight line be drawn from the focus of an ellipse, the eccentricity of which is e, so as to make a given angle β with the tangent; to shew that the locus of its intersection with the tangent will be a circle, which touches or lies entirely without the ellipse as $\cos\beta$ is $<$ or $>$ e.

Let SY be the perpendicular from the focus S upon the tangent PT at P, which cuts the semi-axis major CA, produced, in T. Let Q be a point in PT, such that $\angle SQP = \beta$.



Let $$\angle PTS = \phi, \quad SQ = r, \quad \angle QST = \theta.$$

Then $$r \sin\beta = SY = (a^2 \sin^2\phi + b^2 \cos^2\phi)^{\frac{1}{2}} - ae \sin\phi.$$

Squaring and putting for ϕ its value $\beta - \theta$, we have, for the equation to the locus of Q,
$$r^2 \sin^2\beta + 2aer \sin\beta \sin(\beta - \theta) = a^2(1 - e^2),$$
which is the polar equation to a circle.

If any point of this circle lie in the periphery of the ellipse, (when, intersection being impossible, contact must take place), we have, equating the values of SY in the circle and the ellipse,
$$\frac{a^2(1 - e^2)\, r}{2a - r} = r^2 \sin^2\beta, \qquad a^2(1 - e^2) = (2ar - r^2)\sin^2\beta,$$
$$(a - r)^2 \sin^2\beta = a^2(e^2 - \cos^2\beta).$$

Hence, in order that contact may be possible, $\cos\beta$ must not be greater than e. If $\cos\beta$ be greater than e, the circle lies entirely without the ellipse.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 191.

3. A circle is described about an extremity of the minor axis of an ellipse as centre, with a radius equal to the distance of either directrix from the centre of the ellipse: to prove that this circle will touch the ellipse in two points, one point, or not at all, accordingly as the eccentricity is greater than, equal to, or less than $\frac{1}{\sqrt{2}}$.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ....(1)},$$

or, e being the eccentricity,

$$(1 - e^2)\, x^2 + y^2 = b^2 \text{ ....(2)}.$$

The equation to the circle will be

$$x^2 + (y - b)^2 = \frac{a^2}{e^2},$$

or

$$(1 - e^2)\, x^2 + (1 - e^2)\,(y - b)^2 = \frac{b^2}{e^2} \text{ ....(3)}.$$

Eliminating x^2 between (2) and (3), we have, for all points where the circle and ellipse meet, the equation

$$e^2y^2 + 2\,(1 - e^2)\, by + \frac{b^2}{e^2}\,(1 - e^2)^2 = 0,$$

which, being a perfect square of the equation

$$ey + \frac{b}{e}\,(1 - e^2) = 0 \text{ ....(4)},$$

shews that the curves touch wherever they meet.

From (1) and (4), there is

$$\frac{x^2}{a^2} = 1 - \frac{(1 - e^2)^2}{e^4}.$$

This result shews that $1 - e^2$ must not be greater than e^2, or that e must not be less than $\frac{1}{\sqrt{2}}$. If $e = \frac{1}{\sqrt{2}}$, $x = 0$, and there is contact at the opposite end of the minor axis; if $e > \frac{1}{\sqrt{2}}$, x has two possible values, and accordingly there are two points of contact.

4. To find the equations to the circles which touch the latera recta of an ellipse represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the major axis, and of which the radii are equal to the minor axis.

The required equations are

$$(x - ae \pm b)^2 + y^2 = b^2,$$

and

$$(x + ae \pm b)^2 + y^2 = b^2.$$

5. To find the equations of the circles the diameters of which are the distances between the extremities of the axes of an ellipse defined by the equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

The four circles are defined by the equation

$$x^2 \pm 2x + y^2 \pm 3y = 3.$$

6. If one of the foci of an ellipse be the common focus of two parabolas the vertices of which are at the extremities of the major axis, to prove that these parabolas will intersect at right angles, at points the distance between which is equal to twice the axis minor.

7. If CD be the semi-diameter of an ellipse drawn parallel to a focal chord PSp, to prove that

$$2\,CD^2 = AC.Pp.$$

8. If a circle be described, touching the axis major of an ellipse in one of the foci, and passing through one extremity of the axis minor, to prove that the semi-axis major will be a mean proportional between the diameter of the circle and the semi-axis minor.

9. To find the relation between the central distance of any point of an ellipse and the perpendicular from the centre on the tangent at that point.

If r = the central distance, and p = the perpendicular,

$$p^2 = \frac{a^2b^2}{a^2+b^2-r^2}.$$

10. If two tangents be drawn at the extremities of the axis major of an ellipse, so as to intersect a tangent at any point in two points T, T', and a circle be described upon TT' as a diameter; to prove that this circle will pass through both foci of the ellipse.

11. A parallelogram circumscribes an ellipse, its sides being parallel to conjugate diameters. If 2θ be one of the smaller angles of the parallelogram, when it is equilateral, to prove that its perimeter is equal to $4\sqrt{2}\,.\,a\sec\theta$.

12. A square is described touching an ellipse at the extremities of its minor axis: an ellipse upon the same axis major circumscribes the square. The same process is repeated in relation to the new ellipse, and the operation is continued until there are n ellipses: to prove that, if the eccentricity of the original ellipse be $\left(\frac{n-1}{n}\right)^{\frac{1}{2}}$, the last ellipse will become a circle.

13. If, from the focus S of an ellipse, SQ be drawn, bisecting the angle PSC, (C being the centre of the ellipse and SP any focal distance), to find the locus of Q, supposing SQ to be a mean proportional between SC and SP.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1,$$

the required locus will be an ellipse defined by the equation

$$\frac{(x-ea)^2}{1+e} + \frac{y^2}{1-e} = ea^2.$$

14. If P be any point in an ellipse, the vertex of which is A and nearer focus S, to prove that, if $\angle PAS = \theta$, $\angle ASP = \phi$, and $e =$ the eccentricity,

$$\tan\theta \,.\, \tan\tfrac{1}{2}\phi = 1 + e.$$

15. If γ, θ, be the angles which any diameter of an ellipse makes with its conjugate, and with the axis major, to shew that

$$(a^2 - b^2)\tan\gamma \pm (a^2\tan\theta + b^2\cot\theta) = 0.$$

16. To determine the least distance between the centre C of an ellipse and a point P in its periphery, and, PM being a perpendicular on the axis major, to obtain an expression for the difference between the squares of CP and of the least distance in terms of CM.

The distance is least when P coincides with an extremity B of the semi-axis minor, and

$$(CP)^2 - (CB)^2 = e^2 . (CM)^2.$$

Apollonius: *Conicorum*, lib. v. prop. 11.

17. If PSQ, PHR, be chords drawn from any point P of an ellipse through the foci S and H, ϕ and θ the acute angles which the axis major makes respectively with the line QR produced and with the tangent drawn to the ellipse at the point P, to prove that

$$\tan\phi = \frac{1-e^2}{1+e^2} \,.\, \tan\theta.$$

18. In two confocal ellipses are taken two points P, Q, such that, the ellipses being referred to the major and minor axes as

axes of x and y, their abscissæ are as the major axes of the ellipses: P', Q', are two other such points. To prove that

$$PQ' = P'Q.$$

19. To shew that the equation

$$\left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) . \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right)$$

represents two straight lines, passing through the point (α, β) and touching the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

20. Having given the directions of two systems of conjugate diameters of an ellipse, to determine the directions and relative magnitudes of the two principal axes.

Let θ represent the angle between a principal axis and a given line drawn through the centre of the ellipse: let $2x$ denote the length of this axis, and $2y$ that of the other. Let α, β, be the angles which the two conjugate diameters of one system make with the given line, and α', β', the angles formed with it by the other system.

Then the solution of the problem will be expressed by the following equations,

$$\tan 2\theta = \frac{\cos(\alpha' + \beta') . \cos(\alpha - \beta) - \cos(\alpha + \beta) . \cos(\alpha' - \beta')}{\sin(\alpha' + \beta') . \cos(\alpha - \beta) - \sin(\alpha + \beta) . \cos(\alpha' - \beta')},$$

$$\{-\tan(\theta + \alpha) \tan(\theta + \beta)\}^{\frac{1}{2}} = \frac{y}{x} = \{-\tan(\theta + \alpha') \tan(\theta + \beta')\}^{\frac{1}{2}}.$$

Gergonne: *Annales de Mathématiques*, tom. XII. p. 374.

21. From any point C_2 without a given ellipse, of which the centre is C_1, is drawn a straight line cutting the curve in P_1, P_2. If a point P be taken in C_2P_2 so that C_2P is a mean proportional between C_2P_1, C_2P_2, to find the locus of P.

The equation to the given ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the locus of P will be an ellipse, of which C_2 is the centre, represented by the equation

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1,$$

where α, β, are the coordinates of C_2.

22. If, in the preceding problem, be drawn through C_1, a straight line meeting the given ellipse in Q, and the locus of P in Q_1, Q_2; to prove that C_1Q will be a mean proportional between C_1Q_1, C_1Q_2.

23. If, under the circumstances of the two preceding problems, the given ellipse and the elliptic locus of P intersect in F and G, and if, about FC_1GC_2 be described a third ellipse with axes parallel to those of the given ellipse, and if $(a_{\prime}, b_{\prime})$, $(a_{\prime\prime}, b_{\prime\prime})$, be the semi-axes of the locus of P and of this third ellipse respectively, to prove that

$$\frac{a}{b}=\frac{a_{\prime}}{b_{\prime}}=\frac{a_{\prime\prime}}{b_{\prime\prime}},$$

$$ab+a_{\prime}b_{\prime}=4a_{\prime\prime}b_{\prime\prime};$$

and that the centres of the three ellipses lie in a straight line.

24. If SY, HZ, be perpendiculars from the foci of an ellipse upon the tangent at any point P, to prove that HY, SZ, being joined, will intersect in the normal at the point P, and to find the locus of their intersection.

The equation to the ellipse being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and e denoting its eccentricity, the equation to the required locus will be

$$\frac{x^2}{(1+e^2)^2a^2}+\frac{y^2}{b^2}=\tfrac{1}{4},$$

which is therefore a concentric and similarly placed ellipse, with axes $(1+e^2)\,a$ and b.

25. P is a point in the circumference of an ellipse, S and H its two foci, and A the extremity of the axis major nearest to S. Supposing PH to be increased in length by a small quantity h, the position of P, the direction of PH, the magnitude and position of SP, remaining unchanged: to determine approximately the change in the eccentricity of the ellipse.

If the axis major $= 2a$, $SP = r$, $HP = r'$, $\angle ASP = \theta$, and the eccentricity $= e$; then, approximately,

$$\delta e = \frac{hr}{2ar'} (e + \cos\theta).$$

26. To find the envelop of the perpendiculars drawn through the extremities of all the diameters of a given ellipse.

If the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation to the required envelop will result from the elimination of t between the two equations

$$x^2 = \frac{a^2(t^2 + a^2 - b^2)^2 (t^2 - a^2)}{t^6}, \quad y^2 = \frac{a^2}{b^2} \cdot \frac{\{(2b^2 - a^2)\, t^2 + a^2(a^2 - b^2)\}^2}{t^6}.$$

Roche: *Gergonne, Annales de Mathématiques*, tom. XIV. p. 207.

HYPERBOLA.

SECTION I.

Referred to its Axes. Ordinates.

1. A parabola and an hyperbola have a common axis, the vertex of the former coinciding with one of the foci of the latter: to shew that the product of the distances of the points of intersection from the axis is equal to l^2b^2, l being the latus-rectum of the parabola and b the minor axis of the hyperbola.

The equation to the hyperbola being

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

that to the parabola will be

$$y^2 = l(x - ae):$$

at the intersections

$$y^2 = \frac{b^2}{a^2}\left\{\frac{y^4}{l^2} + \frac{2ae}{l}y^2 + a^2(e^2 - 1)\right\},$$

or
$$y^4 + \ldots\ldots + b^2l^2 = 0:$$

whence the required product, by the theory of equations, is equal to b^2l^2.

2. In two concentric and similarly placed hyperbolæ, are taken two points, the abscissæ of which are as the real axes of the curves. To find the locus of the middle point of the straight line joining the two points.

The equations to the two hyperbolæ being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

the required locus will be an hyperbola represented by the equation

$$\frac{x^2}{(a+a')^2} - \frac{y^2}{(b+b')^2} = \tfrac{1}{4}.$$

SECTION II.

Referred to its Axes. Tangents.

1. The semi-axis CA of an equilateral hyperbola is intersected in T by a tangent to the curve at P: PM is a perpendicular from P upon CA produced: PC is joined. To prove that

$$PC.PM = PT.CM.$$

Let CA, produced indefinitely, be taken as the axis of x, and Cy, at right angles to it, as that of y.

Then, putting $CA = a$, $CM = x$, $PM = y$, and observing that CT is equal to $\frac{a^2}{x}$, we have

$$\begin{aligned}(PT)^2 &= y^2 + \left(x - \frac{a^2}{x}\right)^2\\ &= y^2 + \frac{(x^2-a^2)^2}{x^2}\\ &= y^2 + \frac{y^4}{x^2}\\ &= \frac{y^2}{x^2}(x^2+y^2)\\ &= \frac{(PM)^2}{(CM)^2}\cdot(CP)^2,\end{aligned}$$

or

$$PC.PM = PT.CM.$$

De la Hire: *Sectiones Conicæ*, lib. v. prop. 15.

2. If δ be the perpendicular distance of the centre of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

from the tangent at any point, and if α, β, be the angles which δ

makes with the positive directions of the coordinate axes, to prove that, λ being an arbitrary quantity,

$$a \cos\alpha \left(\frac{1}{\lambda} - \lambda\right) = b \cos\beta \left(\frac{1}{\lambda} + \lambda\right),$$

and
$$\frac{4}{\delta^2} = \frac{1}{a^2}\left(\frac{1}{\lambda} + \lambda\right)^2 + \frac{1}{b^2}\left(\frac{1}{\lambda} - \lambda\right)^2.$$

3. From the centre C of an hyperbola a perpendicular CY is drawn upon the tangent at the point P: to find the coordinates of P, when $PY = CS$.

The coordinates required are

$$x = (e^2 + e - 1)^{\frac{1}{2}} . \frac{a}{e^{\frac{1}{2}}},$$

$$y = (e^2 - 1) . \frac{a}{e^{\frac{1}{2}}}.$$

SECTION III.

Referred to its Axes. Magical Equation to the Tangent.

1. To find the locus of the intersection of a tangent to an hyperbola with a perpendicular upon the tangent from the centre.

The magical equation to the tangent of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is
$$y = \alpha x + (a^2\alpha^2 - b^2)^{\frac{1}{2}} \dots\dots\dots\dots (1).$$

The equation to a perpendicular upon this tangent from the centre is

$$y = -\frac{1}{\alpha} x \dots\dots\dots\dots (2).$$

At the intersection of (1) and (2) we have, α being eliminated between these equations,

$$(x^2 + y^2)^2 = a^2x^2 - b^2y^2,$$

which is the equation to the required locus.

Cor. If the hyperbola be equilateral, the equation becomes

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2),$$

the equation to the lemniscata.

James Bernoulli: *Opera*, tom. I. p. 609.

2. To find the locus of the intersection of a tangent to an hyperbola with a perpendicular upon the tangent from a given point.

The equation to the hyperbola being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and the coordinates of the given point being α, β, the required locus will be defined by the equation

$$\{x (x - \alpha) + y (y - \beta)\}^2 = a^2 (x - \alpha)^2 - b^2 (y - \beta)^2.$$

3. An ellipse and hyperbola having common axes, a pair of tangents is drawn to them, such that the sum of their inclinations to the transverse axis is equal to two right angles: to find the locus of their intersection.

The equations to the ellipse and hpperbola being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

that to the required locus will be

$$4x^2 y^4 = b^4 . (a^2 - x^2).$$

Section IV.

Referred to its Axes. Focal Properties.

1. If, about the exterior focus of an hyperbola, a circle be described with radius equal to the conjugate semi-axis, and tangents be drawn to it from any point in the hyperbola; to prove that the line, joining the points of contact, will touch the circle of which the transverse axis is a diameter.

The equation to the circle, described about the exterior focus, is

$$x^2 + y^2 + 2aex + a^2 = 0.$$

Let (h, k) be any point in the hyperbola: then, tangents being drawn from this point to the circle, the equation to the chord of contact will be

$$(h + ae)\,x + ky + a\,(eh + a) = 0 \;\ldots\ldots\ldots\ldots\; (1).$$

The equation to the hyperbola gives the equation

$$k^2 = (e^2 - 1)\,(h^2 - a^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

The distance of the centre of the hyperbola from (1) is equal to

$$\frac{a\,(eh + a)}{\{(h + ae)^2 + k^2\}^{\frac{1}{2}}}$$

$$= \frac{a\,(eh + a)}{\{(h + ae)^2 + (e^2 - 1)\,(h^2 - a^2)\}^{\frac{1}{2}}}, \text{ from (2),}$$

$$= a.$$

This result establishes the proposition.

2. To find the locus of the centre of a circle inscribed in the triangle *SPH*, *S* and *H* being the foci and *P* any point of an hyperbola.

The equation to the hyperbola being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the required locus is a straight line

$$x = a,$$

which is therefore a tangent at the vertex.

Lardner: *Algebraic Geometry*, p. 128.

3. To find the locus of the centre of a circle, which, the diagram remaining the same as in the preceding problem, touches *SP*, *HP* produced, and *HS* produced.

The required locus is an hyperbola defined by the equation

$$(a+c)^2 y^2 - b^2x^2 = -b^2c^2,$$

$2c$ being equal to SH.

Lardner: *Algebraic Geometry*, p. 128.

4. To find the locus of the centre of a circle which, the diagram remaining the same as in the two preceding problems, touches SH, PS produced, and PH produced.

The equation to the required locus is $x + a = 0$, which is therefore a tangent at the vertex of the opposite branch of the hyperbola.

Lardner: *Algebraic Geometry*, p. 129.

Section V.

Referred to its Axes. Conjugate Diameters. Conjugate Hyperbola.

1. At any point P of an hyperbola, a tangent is drawn, cutting the semi-axis CA in T, and the semi-axis BC, produced, in t: CD is drawn, parallel to tTP, to meet the conjugate hyperbola in D. To prove that

$$PT.Pt = (CD)^2.$$

Let CA, CB, produced indefinitely, be the axes of coordinates.

Let x, y, be the coordinates of P, (x', y') those of D. The equation to the hyperbola being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

that to PTt will be $$\frac{xx_{,}}{a^2} - \frac{yy_{,}}{b^2} = 1.$$

Putting $x_{,} = 0$, we have

$$y_{,} = -\frac{b^2}{y},$$

or $$Ct = \frac{b^2}{y}.$$

Hence, α being the inclination of PTt or DC to CA,

$$PT.\sin\alpha = y,$$

$$Tt.\sin\alpha = Ct = \frac{b^2}{y},$$

$$Pt.\sin\alpha = y + \frac{b^2}{y},$$

and therefore $$PT.Pt.\sin^2\alpha = y^2 + b^2 = \frac{b^2}{a^2}x^2.$$

Again, $$(CD)^2.\sin^2\alpha = y'^2 = \frac{b^2}{a^2}x^2.$$

Hence $$PT.Pt = (CD)^2.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 41.

2. An ellipse and a pair of conjugate hyperbolas are described upon the same axes, and, at the points where any line through the centre meets the ellipse and one of the hyperbolas, tangents are drawn: to find the locus of their intersection.

Let h, k, be the coordinates of the intersection of the two tangents: then, (x, y), (x', y'), being the coordinates of the points of contact with the ellipse and the hyperbola respectively,

$$\left(\frac{hx}{a^2} + \frac{ky}{b^2}\right)^2 = 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad\text{..................} (1),$$

and $$\left(\frac{hx'}{a^2} - \frac{ky'}{b^2}\right)^2 = 1 = \frac{x'^2}{a^2} - \frac{y'^2}{b^2} \quad\text{..................} (2).$$

But $\frac{x}{x'} = \frac{y}{y'}$, by the condition of the problem: hence, from (2), we have

$$\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad\text{..................} (3).$$

From (1) and (3), by addition and subtraction, we have

$$\frac{h^2x^2}{a^4} + \frac{k^2y^2}{b^4} = \frac{x^2}{a^2},$$

and $$2hkx = a^2y$$

Hence, eliminating x and y, we have

$$h^2\left(1 + 4\,\frac{k^4}{b^4}\right) = a^2 \text{.....................} (4),$$

the equation to the required locus.

The equation to the locus resulting from the intersection of the tangent of the ellipse with the corresponding tangent of the conjugate hyperbola is evidently

$$k^2\left(1 + 4\,\frac{h^2}{a^4}\right) = b^2,$$

being derived at once from (4) by interchanging (a, b) and (h, k).

3. To prove that, in a rectangular hyperbola, every diameter is equal to its conjugate diameter.

4. If CP, CD, be conjugate semi-diameters of an hyperbola, to shew that, K being the point of intersection of normals at P, D, KC is perpendicular to PD.

5. To find the area included by the normals to an hyperbola, which pass through the foci of the conjugate hyperbola.

The equation to the hyperbola being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the required area is equal to

$$\frac{2}{a}\,(a^2 + b^2).(a^2 + 2b^2)^{\frac{1}{2}}.$$

SECTION VI.

Referred to its Axes. Asymptotes.

1. In the semi-axis CA of an hyperbola is taken a point M such that, S being the focus,

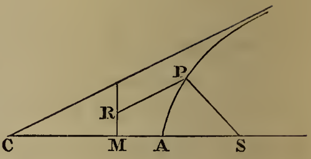
$$CM : CA :: CA : CS;$$

from P, any point in the curve, is drawn PR, parallel to one of the asymptotes, to meet a perpendicular to CA through M

in the point R: to prove that, SP being joined,

$$PR = PS.$$

Let the hyperbola be referred to CA, produced indefinitely, as the axis of x, that of y being at right angles to CA through C. Let $CM = c$. Then the equation to PR, x, y, being the coordinates of P, will be



$$y' - y = \frac{b}{a}(x' - x).$$

Hence, putting c for x', and for y', putting $y_{,} = RM$, we have

$$y_{,} - y = \frac{b}{a}(c - x),$$

and consequently $(PR)^2 = (x - c)^2 + (y - y_{,})^2$

$$= (x - c)^2 . \frac{a^2 + b^2}{a^2}$$

$$= e^2 (x - c)^2,$$

$$PR = e(x - c).$$

But, by hypothesis,

$$a^2 = c.(a^2 + b^2)^{\frac{1}{2}} = cae, \quad a = ce:$$

hence $$PR = ex - a = SP.$$

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 18.

2. If, from a point in an asymptote to two conjugate hyperbolas, a tangent be drawn to each hyperbola, to prove that the points of contact will be in conjugate diameters.

Let (x', y'), (x'', y''), be the points of contact in the hyperbolas represented by the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The equations to the tangents will be

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1, \qquad \frac{yy''}{b^2} - \frac{xx''}{a^2} = 1.$$

At their intersection

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = \frac{yy''}{b^2} - \frac{xx''}{a^2}.$$

But, at their intersection,

$$x : y :: a : b;$$

hence $$\frac{x'}{a} - \frac{y'}{b} = \frac{y''}{b} - \frac{x''}{a} \text{ .... } (1).$$

Now, by the equations to the curves,

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = \frac{y''^2}{b^2} - \frac{x''^2}{a^2}:$$

hence, by (1), $$\frac{x'}{a} + \frac{y'}{b} = \frac{y''}{b} + \frac{x''}{a} \text{ .... } (2).$$

From (1) and (2) we see that

$$x'' = \frac{a}{b} y', \quad y'' = \frac{b}{a} x'.$$

These equations prove the truth of the proposition.

3. If, from a focus of an hyperbola, as centre, a circle be described with diameter equal to the imaginary axis; to prove that it will touch the asymptotes in points where the nearer directrix meets them.

4. A tangent *EPOF* to an hyperbola, at a point *P*, cuts the asymptotes in the points *E*, *F*, and the transverse semi-axis *CA* in *O*: from *F* is drawn the straight line *FKG*, cutting *AC* at right angles in *K*, and the other asymptote *CE* in *G*; from *O*, *A*, are drawn, at right angles to *CA*, *OR*, *AI*, cutting *CE* in *R*, *I*; from *P*, *E*, are drawn *PM*, *EH*, cutting *CA*, produced, at right angles in *M*, *H*: *MP* is produced to meet *CE* in *Q*. To prove that the three lines *GK*, *AI*, *HE*, are in continued proportion: and that the three lines *OR*, *AI*, *MQ*, are also in continued proportion.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 12.

SECTION VII.

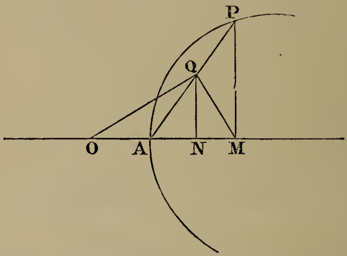
Referred to its Transverse Axis and the Tangent at its Vertex.

1. If any chord AP, through the vertex of an hyperbola, be divided in Q so that

$$AQ : QP :: AC^2 : BC^2,$$

and QM be drawn to the foot of the ordinate MP; to shew that QO, at right angles to QM, cuts the transverse axis in the same ratio.

Draw QN at right angles to the transverse axis, to cut it in N.



Let $AN = x_{,}$, $QN = y_{,}$, $AM = x'$, $PM = y'$, $OA = x$, $\angle QOM = \phi$. Then, from the figure, we see that

$$y_{,} = \tan\phi\,(x_{,} + x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, evidently, by the hypothesis,

$$x_{,} = \frac{a^2}{a^2 + b^2}\,x', \qquad y_{,} = \frac{a^2}{a^2 + b^2}\,y';$$

and therefore

$$\tan\phi = \cot QMN = \frac{x' - x_{,}}{y_{,}} = \frac{(a^2 + b^2)\,x' - a^2x'}{a^2y'} = \frac{b^2x'}{a^2y'}.$$

Hence also, from (1),

$$a^2y' = \frac{b^2x'}{a^2y'}\,\{a^2x' + (a^2 + b^2)\,x\},$$

$$a^2b^2\,(x'^2 + 2ax') = b^2x'\,\{a^2x' + (a^2 + b^2)\,x\},$$

$$2a^3 = (a^2 + b^2)\,x,$$

and therefore $$x = 2a\frac{a^2}{a^2 + b^2}.$$

This result establishes the proposition.

2. The straight line

$$lx + my = \delta$$

is a tangent to the hyperbola

$$y^2 = \frac{b^2}{a^2}(2ax + x^2):$$

to find the value of δ in terms of a, b, l, m.

The required value is given by the equation

$$(\delta + la)^2 = a^2l^2 - b^2m^2.$$

SECTION VIII.

Referred to Conjugate Diameters. Asymptotes.

1. CP is any semi-diameter of an hyperbola, of which C is the centre: a straight line QHK, parallel to PC, cuts the curve in Q and the asymptotes in H, K. To prove that

$$QH.QK = (CP)^2.$$

Let CP, produced indefinitely, be taken as the axis of x, and CD, its conjugate, as that of y. Let $CP = a$, $CD = b$: then the equation to the curve will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Let k represent the constant ordinate of the line QHK: then, O being the intersection of this line with CD, we see, from the equation to the curve, that

$$(OQ)^2 = a^2\left(1 + \frac{k^2}{b^2}\right) \text{..................} (1).$$

Again, at the intersection of the line $y = k$ with the asymptotes, the equations of which are

$$y = \pm\frac{b}{a}x,$$

we have

$$x = \pm\frac{ak}{b};$$

and therefore

$$OH = \frac{ak}{b} = OK.$$

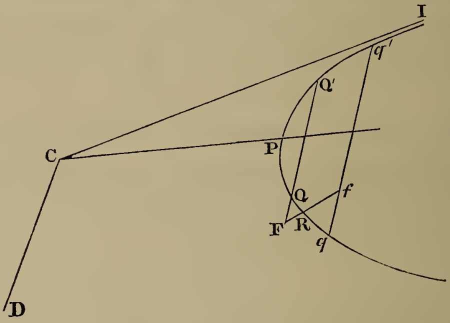
Hence
$$QH.QK = (OQ - OH).(OQ + OK)$$
$$= (OQ)^2 - (OH)^2$$
$$= a^2\left(1 + \frac{k^2}{b^2}\right) - \frac{a^2k^2}{b^2}$$
$$= a^2 = (CP)^2.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 4.

2. QQ', qq', are any two parallel chords of an hyperbola: a straight line FRf, parallel to the asymptote CI, cuts these chords, produced if necessary, in F, f, respectively, and the curve in R. To prove that

$$FQ.FQ' : fq.fq' :: RF : Rf.$$

Let CP be a semi-diameter, conjugate to the two parallel chords: CD the semi-conjugate of CP. Let CP, CD, produced indefinitely, be chosen as axes of x, y, respectively.



If $CP = a'$, $CD = b'$, the equation to the hyperbola is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let h, k, be the coordinates of F: then, writing h for x, and $k - y'$ for y, we shall have a quadratic in y', viz.

$$\frac{h^2}{a'^2} - \frac{(k - y')^2}{b'^2} = 1,$$

the two roots of which are FQ, FQ': thus

$$FQ.FQ' = k^2 + b'^2 - \frac{b'^2}{a'^2}h^2 \quad\ldots\ldots\ldots\ldots\ldots (2).$$

Since Ff is parallel to CI, the equation to which is

$$y = -\frac{b'}{a'}x,$$

the equation to Ff must be of the form

$$y = -\frac{b'}{a'}x + c \text{ .... } (3).$$

But (h, k) is a point in Ff: hence

$$k = -\frac{b'}{a'}h + c.$$

Consequently, from (2),

$$FQ.FQ' = b'^2 + c^2 - 2\frac{b'}{a'}ch \text{ ....} (4).$$

At R, the intersection of (1) and (3),

$$x = \frac{a'}{2b'c}(b'^2 + c^2),$$

and therefore $$x - h = \frac{a'}{2b'c}\left(b'^2 + c^2 - 2\frac{b'}{a'}ch\right).$$

Let this value of $x - h$ be denoted by L: then, by (4),

$$FQ.FQ' = \frac{2b'c}{a'}.L.$$

Similarly, it is evident that, the difference of the abscissæ of R, f, being denoted by l,

$$fq.fq' = \frac{2b'c}{a'}.l.$$

Hence $$\frac{FQ.FQ'}{fq.fq'} = \frac{L}{l}$$

$$= \frac{RF}{Rf}.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 26.

3. From the extremity P of a diameter $P'CP$ of an hyperbola, of which C is the centre, a tangent PE is drawn to meet one of the asymptotes in E: from E is drawn a straight

line EQ, parallel to $P'CP$, to meet the curve in Q: from Q a line QN is drawn, parallel to EP, to meet $P'CP$, produced, in N. To prove that

$$(CP)^2 = PN.P'N.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 8.

4. From the extremity P of any diameter $P'CP$ of an hyperbola, of which C is the centre, a tangent PE is drawn meeting one of its asymptotes in E: through E is drawn EQ, parallel to CP, and meeting the curve in Q: QE is produced to meet the other asymptote in F; through Q is drawn IQJ, parallel to EP, and cutting the asymptotes in I, J. To prove that

$$QE.QF : QI.QJ :: PP' : \text{the parameter of the diameter } PP'.^*$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 9.

5. CP is a semi-diameter of an hyperbola, CD its semi-conjugate: through P is drawn, parallel to CD, a tangent PE, cutting one of the asymptotes in E: QQ' is a chord of the hyperbola, parallel to EP: from any point T in CP is drawn TO, parallel to CE, and cutting $Q'Q$, produced if necessary, in O: from Q is drawn QR, parallel to EC, cutting CP in R. To prove that

$$OQ.OQ' : \text{quadrilateral } QRTO :: (PE)^2 : \text{triangle } PEC.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 24.

SECTION IX.

Referred to Conjugate Diameters. Conjugate Hyperbola.

1. From any points Q, q, in the conjugate hyperbola, parallel straight lines QRR', qrr', are drawn to intersect the hyperbola in the points R, R', and r, r', respectively.

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* The name *parameter* was given by De la Hire to a third proportional to any diameter $2a'$ and its conjugate $2b'$; the parameter being thus equal to $\frac{2b'^2}{a'}$. Apollonius had called such a line the *latus rectum* of the diameter.

To prove that $QR.QR' = qr.qr'$.

Let the hyperbola be referred to CP, CD, produced indefinitely, as axes of coordinates, CP being a semi-diameter conjugate to RR', or rr', and CD being a semi-diameter conjugate to CP.

Then the equation to the hyperbola is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \quad\text{.........................}(1),$$

and to the conjugate hyperbola is

$$\frac{y^2}{b'^2} - \frac{x^2}{a'^2} = 1 \quad\text{.........................}(2).$$

Let h, k, be the coordinates of Q: then, putting in (1), h for x, and $k - y'$ for y, we shall have

$$\frac{h^2}{a'^2} - \frac{(k-y')^2}{b'^2} = 1,$$

the two values of y' in this equation being QR, QR'. Hence

$$QR.QR' = b'^2\left(\frac{k^2}{b'^2} - \frac{h^2}{a'^2} + 1\right).$$

But h, k, being a point in (2),

$$\frac{k^2}{b'^2} - \frac{h^2}{a'^2} = 1:$$

hence $QR.QR' = 2b'^2$.

Similarly $qr.qr' = 2b'^2$.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 34.

2. To find the locus of the middle points of a system of parallel chords drawn between an hyperbola and the conjugate hyperbola.

Let two conjugate diameters be taken as axes of coordinates, the axis of y being parallel to the chords.

Let y_1, y_2, be the ordinates of the two ends of a chord of which x is the abscissa. Then, a, b, representing the magnitudes of the conjugate semi-diameters,

$$\frac{x^2}{a^2} - \frac{y_1^{\ 2}}{b^2} = 1 \text{ .... } (1),$$

$$\frac{y_2^{\ 2}}{b^2} - \frac{x^2}{a^2} = 1 \text{ .... } (2).$$

Let y be the ordinate of the middle point of the chord: then

$$y_1 + y_2 = 2y \text{ .... } (3).$$

Adding (1) and (2), and paying attention to (3), we get

$$y_2 - y_1 = \frac{b^2}{y} \text{ .... } (4).$$

Squaring (3) and (4), and then adding, we have

$$y_1^{\ 2} + y_2^{\ 2} = 2y^2 + \frac{b^4}{2y^2} \text{ .... } (5).$$

But, subtracting (1) from (2), we see that

$$\frac{y_1^{\ 2} + y_2^{\ 2}}{b^2} = \frac{2x^2}{a^2} \text{ .... } (6).$$

From (5) and (6), there is

$$\frac{2y^2}{b^2} + \frac{b^2}{2y^2} = \frac{2x^2}{a^2},$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{b^2}{4y^2},$$

which is the equation to the required locus.

3. *CP*, *CP'*, are two semi-diameters of an hyperbola; *CD*, *CD'*, are two semi-diameters of the conjugate hyperbola, respectively conjugate to *CP*, *CP'*: from *P* are drawn *PE*, *PF*, parallel respectively to *P'C*, *CD'*, and cutting *CD*, or *CD* produced, in *E*, *F*. To prove that

$$CE.CF = (CD)^2.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 38.

4. CP, CP', are two semi-diameters of an hyperbola; CD, CD', are two semi-diameters of the conjugate hyperbola, respectively conjugate to CP, CP': from P is drawn PT, parallel to DC, and cutting CP' in T; and from D is drawn DK, parallel to $D'C$, and cutting CP in K. To prove that

$$\text{triangle } CPT = \text{triangle } CDK.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 39.

SECTION X.

Referred to any two Diameters. Conjugate Hyperbola.

1. The equation to an hyperbola being

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} + \frac{2xy}{mn} = 1,$$

to find the equation to the conjugate hyperbola?

The equations to the two curves referred to their principal axes are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

If we change the directions of their axes of coordinates, by substituting $\lambda x + \mu y$, $\lambda' x + \mu' y$, for x, y, respectively, it is plain that the left-hand member of either of the equations will still remain the same as that of the other.

Hence the equation to one being

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} + \frac{2xy}{mn} = 1,$$

that to the other will be

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} + \frac{2xy}{mn} = -1.$$

2. CP, CP', are two semi-diameters not conjugate, the former of an hyperbola, the latter of its conjugate hyperbola. A tangent at P intersects CP', produced if necessary, in T, and a tangent at P' intersects CP, produced if necessary, in T'. To prove that the triangle PCT is equal to the triangle $P'CT'$.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 36.

SECTION XI.

Referred to its Asymptotes.

1. A straight line touches an hyperbola in P and cuts its asymptotes in H, K; to prove that HK is the least possible when P coincides with the extremity of the transverse axis.

The equation to the hyperbola, referred to its asymptotes as axes, is

$$xy = c^2.$$

Let x, y, be the coordinates of the point P: the equation to the tangent HK will be

$$xx' + yy' = 2c^2.$$

Hence, C being the centre of the hyperbola,

$$CH = \frac{2c^2}{x}, \quad CK = \frac{2c^2}{y},$$

and therefore, ω denoting the angle between the asymptotes,

$$(HK)^2 = 4c^4 \left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{2\cos\omega}{xy}\right)$$

$$= \frac{4c^4}{x^2y^2} (x^2 + y^2 - 2xy\cos\omega)$$

$$= 4c^2 (x^2 + y^2 - 2c^2\cos\omega).$$

Let H', K', be the positions of H, K, when P is at the extremity A of the transverse axis: then, since CA bisects the angle between the asymptotes, the coordinates of A are c, c;

hence
$$(H'K')^2 = 4c^2\,(2c^2 - 2c^2\cos\omega),$$
$$\begin{aligned}(HK)^2 - (H'K')^2 &= 4c^2\,(x^2 + y^2 - 2c^2)\\ &= \frac{4c^2}{x^2}\,(x^4 + x^2y^2 - 2c^2x^2)\\ &= \frac{4c^2}{x^2}\,(x^2 - c^2)^2.\end{aligned}$$

This result shews that $H'K'$ is less than HK; which establishes the proposition.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 33.

2. A straight line $HPQK$ cuts the asymptotes Cx, Cy, of an hyperbola in the points H, K, and the curve in P, Q: to prove that
$$PH = QK.$$

The equation to the hyperbola is
$$xy = c^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1):$$
let that to the straight line be
$$\frac{x}{a} + \frac{y}{b} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

At the intersections of (1) and (2),
$$x^2 - ax + \frac{ac^2}{b} = 0.$$

Hence, (x_1, y_1) being the coordinates of P, and (x_2, y_2) those of Q,
$$x_1 + x_2 = a$$
and
$$\left.\begin{aligned}x_1 + x_2 &= a\\ y_1 + y_2 &= b\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Consequently, ω being the angle between the asymptotes,
$$\begin{aligned}(PH)^2 &= (a - x_1)^2 + y_1^2 - 2\,(a - x_1)\,y_1\cos\omega\\ &= (b - y_2)^2 + x_2^2 - 2\,(b - y_2)\,x_2\cos\omega, \quad \text{by (3)},\\ &= (QK)^2,\end{aligned}$$
or
$$PH = QK.$$

Ἐὰν ὑπερβολῇ εὐθεῖα συμπίπτῃ κατὰ δύο σημεῖα· ἐκβαλλομένη ἐφ᾽ ἑκάτερα συμπεσεῖται ἀσυμπτώτοις, καὶ αἱ ἀπολαμβανόμεναι ἀπ᾽ αὐτῆς ὑπὸ τῆς τομῆς πρὸς ταῖς ἀσυμπτώτοις ἴσαι ἔσονται.

ἈΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ *Κωνικῶν τὸ δεύτερον· Πρότασις ή.*

3. A tangent to the curve $xy = c^2$ is intersected by a perpendicular upon it from the origin: prove that, if ω be the angle between the coordinate axes, the equation to the locus of the intersection will be

$$y\,(x\cos\omega + y) + x(y\cos\omega + x) = 2c\{(x\cos\omega + y)\,(y\cos\omega + x)\}^{\frac{1}{2}}.$$

Let the equation to the tangent be

$$y = \alpha x + \beta:$$

then the roots of the equation

$$\alpha x^2 + \beta x - c^2 = 0$$

must be equal: hence $\beta^2 = -4\alpha c^2$,

and the equation to the tangent will become

$$y = \alpha x + 2c\,(-\alpha)^{\frac{1}{2}} \dots\dots\dots\dots(1).$$

Let the equation to the perpendicular upon it be

$$y = \alpha' x \dots\dots\dots\dots (2).$$

Then, by the condition of perpendicularity,

$$1 + \alpha\alpha' + (\alpha + \alpha')\cos\omega = 0 \dots\dots\dots\dots(3).$$

From (2) and (3) there is

$$x + \alpha y + (\alpha x + y)\cos\omega = 0,$$

$$x + y\cos\omega = -\alpha\,(y + x\cos\omega),$$

and therefore, from (1), we get for the equation to the locus,

$$y\,(y + x\cos\omega) + x\,(x + y\cos\omega) = 2c\,\{(x + y\cos\omega).(y + x\cos\omega)\}^{\frac{1}{2}}.$$

4. In every triangle inscribed in an equilateral hyperbola, the point of concourse of the perpendiculars, drawn from the angles to the opposite sides, lies in the curve.

Let the coordinates of the three angular points of any inscribed triangle be (x, y), (x', y'), (x'', y''). Then the equation to the side joining (x, y), (x', y'), will be

$$y_1 - y = \frac{y' - y}{x' - x}\,(x_1 - x) \dots\dots\dots\dots(1).$$

But, by the equation to the hyperbola, supposing the asymptotes to be the axes of coordinates,

$$xy = c^2, \quad x'y' = c^2,$$

and therefore

$$\frac{y'-y}{x'-x} = -\frac{c^2}{xx'}.$$

The equation (1) therefore becomes

$$y_1 - y = -\frac{c^2}{xx'}(x_1 - x).$$

The equation to the perpendicular to this line through (x'', y'') will be

$$y_1 - y'' = \frac{xx'}{c^2}(x_1 - x'') \dots\dots\dots\dots\dots\dots (2).$$

Similarly, the equation to the perpendicular through the angular point (x, y) will be

$$y_1 - y = \frac{x'x''}{c^2}(x_1 - x) \dots\dots\dots\dots\dots\dots (3).$$

At the intersection of (2) and (3), we have, multiplying (2) by x'', (3) by x, subtracting, and observing that $x''y''$, xy, are both equal to c^2,

$$y_1 = -\frac{xx'x''}{c^2}.$$

Similarly, we must have

$$x_1 = -\frac{yy'y''}{c^2}.$$

By symmetry it is plain that these are the coordinates of a point through which the three perpendiculars pass. Moreover

$$x_1 y_1 = \frac{xy}{c^2} \cdot \frac{x'y'}{c^2} \cdot \frac{x''y''}{c^2} \cdot c^2$$
$$= c^2:$$

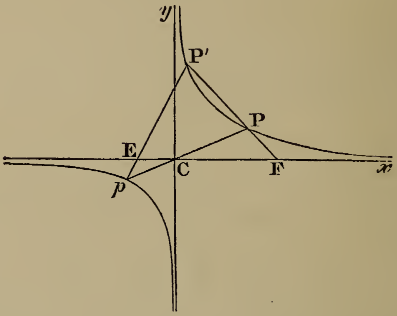
hence (x_1, y_1) are the coordinates of a point in the curve.

Brianchon, Poncelet: *Gergonne*, *Annales de Mathématiques*, tom. XI. p. 205.

5. To prove that the straight lines drawn from any point in an equilateral hyperbola to the extremities of any diameter, are inclined at equal angles to the asymptotes.

The equation to the hyperbola, referred to its asymptotes Cx, Cy, (which constitute a system of rectangular axes) will be

$$xy = c^2.$$



Let PCp be any diameter; (x, y) being the coordinates of P, $(-x, -y)$ of p, and (x', y') of any point P' in the curve.

Let $\angle P'Ex = \theta$, and $\angle P'Fx = \phi$. Then

$$\tan\phi = \frac{y' - y}{x' - x} = c^2 \cdot \frac{\frac{1}{x'} - \frac{1}{x}}{x' - x} = -\frac{c^2}{xx'}.$$

Putting $-x$ for x, and θ for ϕ, we have also

$$\tan\theta = \frac{c^2}{xx'}.$$

From these results it appears that

$$\theta = \pi - \phi,$$

and therefore $\angle P'EF = \angle P'FE.$

Cor. It may hence be easily inferred that, if from another point P'', lines be drawn to P and p, $\angle P'PP'' = \angle P'pP''$.

Newton: *Arithmetica Universalis*, prob. 35.

Gergonne: *Annales de Mathématiques*, tom. II. pp. 32, 126.

6. If any hexagon be circumscribed about an hyperbola, the three diagonals, joining opposite angles, will all pass through one point.

The equation to the hyperbola being

$$4xy = c^2,$$

let the equation to a tangent be

$$\frac{x}{\alpha_1} + \frac{y}{\beta_1} = 1:$$

then the roots of the equation

$$\frac{x^2}{\alpha_1} - x + \frac{c^2}{4\beta_1} = 0,$$

which belong to the common point of the tangent and curve, must be equal: hence $\alpha_1\beta_1 = c^2$.

Thus the equation to the tangent is

$$\frac{x}{\alpha_1} + \frac{\alpha_1 y}{c^2} = 1.$$

At the intersection of this tangent and another, represented by the equation

$$\frac{x}{\alpha_2} + \frac{\alpha_2 y}{c^2} = 1,$$

we shall have

$$x_{1,2} = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}, \quad y_{1,2} = \frac{c^2}{\alpha_1 + \alpha_2}.$$

These will be the coordinates of one of the angles of the circumscribed hexagon, the two tangents being two of the sides. We shall have analogous expressions for the coordinates of the other angles.

The equation to the diagonal through the angles $(x_{1,2}, y_{1,2})$, $(x_{4,5}, y_{4,5})$, will be

$$x(y_{4,5} - y_{1,2}) - y(x_{4,5} - x_{1,2}) = x_{1,2}y_{4,5} - x_{4,5}y_{1,2};$$

or, if we substitute for the coordinates of the angular points, their values,

$$c^2x\{(\alpha_4 - \alpha_1) - (\alpha_2 - \alpha_5)\} + y\{\alpha_4\alpha_1(\alpha_5 - \alpha_2) + \alpha_5\alpha_2(\alpha_4 - \alpha_1)\}$$
$$= c^2(\alpha_4\alpha_5 - \alpha_1\alpha_2)\ldots\ldots(1).$$

Similarly, the equation to the diagonal, through the angular points $(x_{2,3}, y_{2,3})$, $(x_{5,6}, y_{5,6})$, will be

$$c^2x\{(\alpha_5 - \alpha_2) - (\alpha_3 - \alpha_6)\} + y\{\alpha_5\alpha_2(\alpha_6 - \alpha_3) + \alpha_6\alpha_3(\alpha_5 - \alpha_2)\}$$
$$= c^2(\alpha_5\alpha_6 - \alpha_2\alpha_3)\ldots\ldots(2).$$

At the intersection of these two diagonals, multiplying the equation (1) by $\alpha_3 - \alpha_6$, and the equation (2) by $\alpha_4 - \alpha_1$, adding the latter of the resulting equations to the former, and dividing the final equation by $\alpha_2 - \alpha_5$, we shall get

$$c^2 x \{(\alpha_6 - \alpha_3) - (\alpha_4 - \alpha_1)\} + y\{\alpha_6\alpha_3(\alpha_1 - \alpha_4) + \alpha_1\alpha_4(\alpha_6 - \alpha_3)\}$$
$$= c^2(\alpha_6\alpha_1 - \alpha_3\alpha_4) \ldots\ldots (3).$$

But, as is evident from symmetry, the equation (3) belongs to the third diagonal, namely, that which passes through the points $(x_{3,4}, y_{3,4})$, $(x_{6,1}, y_{6,1})$. Thus we see that the two diagonals (1) and (2) intersect in the third; which establishes the theorem.

COR. Subtracting the sum of (1) and (3) from (2), we get for the value of y, at the point through which the three diagonals pass,

$$y\{\alpha_1\alpha_2(\alpha_4 + \alpha_5) - \alpha_2\alpha_3(\alpha_5 + \alpha_6) + \alpha_3\alpha_4(\alpha_6 + \alpha_1) - \alpha_4\alpha_5(\alpha_1 + \alpha_2)$$
$$+ \alpha_5\alpha_6(\alpha_2 + \alpha_3) - \alpha_6\alpha_1(\alpha_3 + \alpha_4)\}$$
$$= c^2(\alpha_1\alpha_2 - \alpha_2\alpha_3 + \alpha_3\alpha_4 - \alpha_4\alpha_5 + \alpha_5\alpha_6 - \alpha_6\alpha_1).$$

The values of x will, as symmetry points out, be obtained by writing, in this equation, x for y, and β for α, throughout.

Cambridge Mathematical Journal, vol. IV. p. 164.

7. To determine the hyperbola which has two given lines as asymptotes and which passes through a given point.

Let h, k, be the coordinates of the given point, referred to the two given lines as axes of coordinates.

The equation to the hyperbola is

$$xy = hk.$$

Δύο δοθεισῶν εὐθειῶν γωνίαν περιεχουσῶν, καὶ σημείου ἐντὸς τῆς γωνίας· γράψαι διὰ τοῦ σημείου κώνου τομὴν τὴν καλουμένην ῾υπερβολὴν, ὥστε ἀσυμπτώτους εἶναι τὰς δοθείσας εὐθείας.

'ΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ *Κωνικῶν τὸ δεύτερον· Πρότασις δ'.*

Pappus: *Mathematicæ Collectiones à Commandino*, lib. VII. prob. 15, prop. CCIV., p. 277.

8. To find the locus of the intersection of a pair of tangents to a rectangular hyperbola $xy = c^2$, which are such that the product of the tangents of their inclinations to the axis of x is constant.

If k denote the constant product, the required locus will be a straight line the equation of which is $y = kx$.

9. A straight line moves in the plane of a given angle, so as to form with the sides of the angle a triangle of given area: to find the equation to the locus of the centre of gravity of the triangle.

The two sides of the angle being taken as the axes of coordinates, the equation to the curve will be

$$xy = \frac{2k^2}{9 \sin \alpha},$$

where k denotes the given area, and α the given angle.

Francœur: *Gergonne*, *Annales de Mathématiques*, tom. x. p. 79.
Gergonne: *Annales de Mathématiques*, tom. x. p. 87.

10. If a concentric circle cut a rectangular hyperbola, referred to its asymptotes as axes, and PT, PT', be tangents to the circle and hyperbola respectively, meeting the axis of x in T, T', PM being the ordinate at the point of intersection: to prove that

$$MT.MT' = MP^2.$$

11. To find the general form of the polar equation to the tangent of an equilateral hyperbola, an asymptote being the prime radius vector.

The equation to the hyperbola being $xy = c^2$, the polar equation to the tangent will be

$$r\left(\alpha \cos\theta + \frac{\sin\theta}{\alpha}\right) = 2c,$$

where α is an arbitrary constant.

12. To prove that the secants, drawn from any one of the points of an hyperbola to two fixed points in the curve, always

intercept, on either asymptote, a constant length equal to the length intercepted on the same asymptote between the secant through the two fixed points and the tangent at either of them.

Sturm, Vecten, Querret: *Gergonne, Annales de Mathématiques*, tom. XV. p. 100.

Quetelet: *Corr. Math. et Phys.*, tom. II. p. 323.

13. To prove that every chord of an hyperbola divides into two equal parts the portion of either asymptote which is included between the tangents at its extremities.

Sturm, Vecten, Querret: *Gergonne, Annales de Mathématiques*, tom. XV. p. 102.

14. If any two tangents be drawn to an hyperbola, and lines be drawn joining the points in which they intersect the asymptotes, to prove that these lines will be parallel to one another.

15. To find the equation to the conjugate diameter of any system of parallel chords in an hyperbola.

If the equation to any one of the chords be

$$\frac{x}{m} + \frac{y}{n} = 1,$$

the equation to their conjugate diameter will be

$$\frac{x}{m} = \frac{y}{n}.$$

16. From any fixed point P in an hyperbola, are drawn PH, PK, parallel to the asymptotes; and, from another fixed point Q in the hyperbola, is drawn any straight line cutting PH, PK, in M, N, respectively, and the curve in R: to prove that

$$RM \propto RN.$$

17. If a pair of conjugate diameters of an ellipse be, when produced, asymptotes to an hyperbola, to prove that the points of the hyperbola at which tangents to the hyperbola will also be

tangents to the ellipse, lie in an ellipse concentric and similar in form to the given one.

18. A tangent at a point E of an hyperbola intersects one of the asymptotes in F: an indefinite line EN is drawn through E parallel to this asymptote: from F any line $FGHI$ is drawn, cutting the curve in G, I, and the line EN in H: to prove that the line FI is harmonically divided in the points G, H.

De la Hire: *Sectiones Conicæ*, lib. II. prop. 22.

19. Cx, Cy, are the asymptotes of an hyperbola: HK is any chord: CP is a semi-diameter, conjugate to HK: from K, P, H, are drawn, parallel to yC, the three lines KN, PI, HM, intersecting Cx in N, I, M. To prove that these three lines are continued proportionals.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 15.

20. If the abscissæ of any number of points in an hyperbola, referred to its asymptotes, are in arithmetical, to shew that the ordinates are in harmonical progression.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 16.

21. From any two points P, P', in an hyperbola are drawn two straight lines, PM, $P'M'$, parallel to the asymptote yC, and meeting the asymptote Cx in M, M': from the same two points are drawn also PN, $P'N'$, parallel to xC, and cutting Cy in N, N'. To prove that the mixtilineal area $PNN'P'$ is equal to the mixtilineal area $PMM'P'$.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 17.

22. A tangent is drawn at a point P of an hyperbola, cutting the asymptote Cy in E: from E is drawn any straight line EKH cutting one branch of the hyperbola in the points K, H: Kk, PM, Hh, are drawn, parallel to yC, to cut the other asymptote Cx in the points k, M, h. To prove that

$$Hh + Kk = 2PM.$$

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 19.

23. At a point P of an hyperbola is drawn a tangent PM, cutting one of the asymptotes in M. From a point Q in the curve is drawn a line QN, parallel to PM, cutting the same asymptote in N: from Q is drawn also QE, parallel to NC, cutting the semi-diameter CP in the point E. To prove that the triangle PCM is equal to the quadrilateral $CEQN$.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 20.

24. CP is any semi-diameter of an hyperbola, CD is its semi-conjugate. At P is drawn, parallel to DC, a tangent PE, cutting one of the asymptotes in E: in CE, produced, is taken any point H, and from H is drawn, parallel to CD, a straight line HQ, Q being its first intersection with the curve: from Q is drawn the straight line QI, parallel to PC and meeting CE, produced if necessary, in I; from Q is drawn also QF, parallel to EC, and meeting CD in F. To prove that the quadrilateral $CIQF$ is equal to the triangle CEP.

De la Hire: *Sectiones Conicæ*, lib. IV. prop. 21.

25. If any right-angled triangle be inscribed in an equilateral hyperbola, the perpendicular let fall from the summit of the right angle upon the hypotenuse is a tangent to the curve.

Brianchon, Poncelet: *Gergonne, Annales de Mathématiques*, tom. XI. p. 206.

26. If, on a chord of an hyperbola, considered as a diagonal, be constructed a parallelogram the sides of which are respectively parallel to the asymptotes of the curve, to prove that the other diagonal of this parallelogram, produced if necessary, will pass through the centre of the curve.

Sturm, Vecten, Querret: *Gergonne, Annales de Mathématiques*, tom. XV. p. 102.

27. If, on the three sides of a triangle, taken in turn as diagonals, be constructed parallelograms, the sides of which are

respectively parallel to two given straight lines, to prove that the three other diagonals of these parallelograms will pass through the centre of an hyperbola which, being circumscribed about the triangle, has its asymptotes parallel to the two given straight lines.

Sturm, Vecten, Querret: *Gergonne, Annales de Mathématiques*, tom. XV. p. 103.

28. An hyperbola, denoted by the equation $xy = c^2$, is intersected by a parabola, the equation of which is

$$y^2 = ax + \beta:$$

to prove that, y_1, y_2, being the lengths of the ordinates of the intersections of one branch, and y_3 the length of the ordinate of the intersection of the other branch,

$$y_3 = y_1 + y_2.$$

De la Hire: *Sectiones Conicæ*, lib. V. prop. 32.

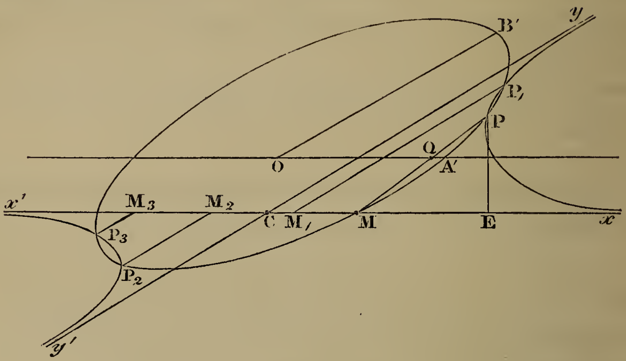
29. From a point P, in one of the branches of a rectangular hyperbola, $xy = c^2$, a perpendicular PM is drawn to one of the asymptotes: PM is bisected in Q, and through Q is drawn, perpendicular to the other asymptote, an indefinite straight line: in this indefinite line is taken any point O; OP is joined. A circle is described, of which O is the centre and OP the radius. To prove that, if the circle cut the locus of $xy = c^2$ in four points, the same branch will be intersected in another point P_1, and the other branch in two points P_2, P_3, and that, y_1, y_2, y_3, denoting the magnitudes of the ordinates of P_1, P_2, P_3,

$$y_2 + y_3 = y_1.$$

De la Hire: *Sectiones Conicæ*, lib. V. prop. 33.

30. From any point P in one branch of any hyperbola, is drawn PM, parallel to the asymptote yCy', and cutting the asymptote xCx' in M. PM is bisected in Q, and, in an indefinite straight line through Q, parallel to Cx, is taken some point O. An ellipse is described, passing through P and M, of which O is

the centre, and OA', OB', two conjugate diameters parallel to Cx, Cy, respectively, such that $OA' : OB' :: PM : PE$, PE being at right angles to Cx.



To prove that, if the ellipse intersects the two branches in P_1, P_2, P_3, then, P_1M_1, P_2M_2, P_3M_3, being parallel to Cy,

$$P_1M_1 = P_2M_2 + P_3M_3.$$

De la Hire: *Sectiones Conicæ*, lib. v. prop. 34.

SECTION XII.

Referred to any Rectangular Axes.

1. To prove that, in the equation to a rectangular hyperbola, referred to any rectangular axes whatever, the coefficients of x^2 and y^2 are equal with opposite signs.

The equation to a rectangular hyperbola referred to its principal axes is $$x^2 - y^2 = a^2.$$

Turning the axes through an angle θ, we shall change this equation into

$$(x \cos\theta - y \sin\theta)^2 - (x \sin\theta + y \cos\theta)^2 = a^2,$$

or $$(x^2 - y^2) \cos 2\theta - 2xy \sin 2\theta = a^2.$$

If we transfer the origin to any new place, the coefficients of x^2 and y^2 will not be affected. The proposition is therefore established.

2. If two points be taken in each of two rectangular axes so as to satisfy the condition that a rectangular hyperbola may pass through all the four, shew that the position of the hyperbola is indeterminate, and that its centre describes a circle which passes through the origin and bisects all the lines which join the points two and two.

Let A, A', be two points taken in the axis of x, and B, B', two points in that of y. Let $OA=\alpha$, $OA'=\alpha'$, $OB=\beta$, $OB'=\beta'$. The equation to any conic section passing through these four points will be

$$\frac{x^2}{\alpha\alpha'} - bxy + \frac{y^2}{\beta\beta'} - \frac{x}{\alpha\alpha'}(\alpha+\alpha') - \frac{y}{\beta\beta'}(\beta+\beta') + 1 = 0 \ldots (1).$$

The hyperbola being rectangular, we must have

$$\alpha\alpha' + \beta\beta' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since b is independent of α, α', β, β', the position of the hyperbola is indeterminate.

Suppose that the origin is changed to a point X, Y, the directions of the axes not being altered. Then (1) will become

$$\frac{(x'+X)^2}{\alpha\alpha'} - b(x'+X)(y'+Y) + \frac{(y'+Y)^2}{\beta\beta'} - \frac{x'+X}{\alpha\alpha'}(\alpha+\alpha') - \frac{y'+Y}{\beta\beta'}(\beta+\beta') + 1 = 0.$$

If X, Y, be the coordinates of the centre of the hyperbola,

$$\frac{2X}{\alpha\alpha'} - bY - \frac{\alpha+\alpha'}{\alpha\alpha'} = 0, \quad \frac{2Y}{\beta\beta'} - bX - \frac{\beta+\beta'}{\beta\beta'} = 0,$$

and therefore, eliminating b, and paying attention to equation (2),

$$2(X^2+Y^2) - (\alpha+\alpha')X - (\beta+\beta')Y = 0,$$

$$\text{or } \left(X - \frac{\alpha+\alpha'}{4}\right)^2 + \left(Y - \frac{\beta+\beta'}{4}\right)^2 = \left(\frac{\alpha+\alpha'}{4}\right)^2 + \left(\frac{\beta+\beta'}{4}\right)^2 \ldots (3),$$

the equation to a circle passing through O.

The coordinates of the middle points of the lines AB, $A'B$, AB', $A'B'$, are $(\frac{1}{2}\alpha, \frac{1}{2}\beta)$, $(\frac{1}{2}\alpha', \frac{1}{2}\beta)$, $(\frac{1}{2}\alpha, \frac{1}{2}\beta')$, $(\frac{1}{2}\alpha', \frac{1}{2}\beta')$: these coordinates, by virtue of (2), satisfy the equation (3). This shews that these lines are all bisected by the circle.

3. If $uv = c^2$ be the equation to an hyperbola, where

$$u = x\cos\alpha + y\sin\alpha - h,$$
$$v = x\cos\beta + y\sin\beta - k,$$

and c is constant, to prove that

$$uv = 0, \quad u^2 = v^2, \quad u^2 = m^2v^2,$$

are the respective equations to the asymptotes, axes, and a pair of conjugate diameters, m being an arbitrary constant.

SECTION XIII.

Referred to any Rectangular Axes. Reduction.

The equation to an hyperbola referred to any axes whatever is

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

a, b, c, being subject to the condition

$$c^2 > ab.$$

The process to be adopted for the purpose of finding the centre as well as the magnitudes and positions of the axes is the same as in the reduction of the general equation to the ellipse.

1. To determine the positions of the axes of the curve

$$xy - x^2\tan\alpha = \tfrac{1}{4}b^2\cot\alpha,$$

and the ratio between them.

Putting, in the equation,

$$x = x'\cos\theta - y'\sin\theta, \quad y = x'\sin\theta + y'\cos\theta,$$

we have

$$(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) - (x'\cos\theta - y'\sin\theta)^2.\tan\alpha = \tfrac{1}{4}b^2\cot\alpha.$$

Equating to zero the coefficient of $x'y'$, we get

$$\cos 2\theta + \tan\alpha\sin 2\theta = 0,$$
$$\tan 2\theta = -\cot\alpha = \tan(\pm\tfrac{1}{2}\pi + \alpha),$$
$$2\theta = \pm\tfrac{1}{2}\pi + \alpha,$$

which shews that the axes are inclined at angles $\frac{1}{4}(\pi + 2\alpha)$, $\frac{1}{4}(-\pi + 2\alpha)$, to the axis of x.

Taking the former axis as that of x', the equation becomes

$$(x'^2 - y'^2)\sin 2\theta - \tan\alpha\,\{x'^2(1+\cos 2\theta) + y'^2(1-\cos 2\theta)\} = \tfrac{1}{4}b^2\cot\alpha,$$

$$x'^2(\cos\alpha - \tan\alpha + \sin\alpha\tan\alpha) - y'^2(\cos\alpha + \tan\alpha + \sin\alpha\tan\alpha) = \tfrac{1}{2}b^2\cot\alpha,$$

$$x'^2(1-\sin\alpha) - y'^2(1+\sin\alpha) = \tfrac{1}{2}b^2\frac{\cos^2\alpha}{\sin\alpha},$$

$$\frac{x'^2}{1+\sin\alpha} - \frac{y'^2}{1-\sin\alpha} = \tfrac{1}{2}b^2\operatorname{cosec}\alpha,$$

$$\frac{x'^2}{(\cos\frac{1}{2}\alpha + \sin\frac{1}{2}\alpha)^2} - \frac{y'^2}{(\cos\frac{1}{2}\alpha - \sin\frac{1}{2}\alpha)^2} = \tfrac{1}{2}b^2\operatorname{cosec}\alpha.$$

The ratio between the axes is therefore equal to

$$\frac{1-\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\alpha} = \tan\frac{\pi - 2\alpha}{4}.$$

2. To prove that the equation

$$4xy - 3x^2 = a^2$$

will be changed into the equation

$$x^2 - 4y^2 = a^2,$$

if the axes of coordinates be turned through an angle the tangent of which is 2.

3. The equation to an hyperbola being

$$y = x + \frac{a^2}{x},$$

to find its equation when it is referred to its axes.

The required equation is

$$x^2\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) - y^2\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) = a^2.$$

4. The equation to an hyperbola being

$$x^2 - y^2 + 2xy - x + y - 1 = 0,$$

to find its equation when referred to its axes.

The required equation is

$$x^2 - y^2 = \frac{3}{4\sqrt{2}}.$$

Garnier: *Géométrie Analytique*, p. 131.

SECTION XIV.

Polar Equation. Centre the Pole.

1. To prove that of all diameters of an hyperbola the transverse axis is the least.

The polar equation to the hyperbola, the centre being the pole and the transverse semi-axis being the prime radius vector, is

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2},$$

whence
$$\frac{1}{a^2} - \frac{1}{r^2} = \sin^2\theta\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

The expression for $\frac{1}{a^2} - \frac{1}{r^2}$ being essentially positive, it is evident that $\frac{1}{r^2}$ is less than $\frac{1}{a^2}$, or a^2 less than r^2, or a than r, a conclusion which establishes the proposition.

De la Hire: *Sectiones Conicæ*, lib. II. prop. 36.

2. An hyperbola, of which C is the centre, is cut in two points P, P', by a given straight line: to find the tangent of the sum of the inclinations of CP, CP', to the transverse axis.

The equations to the hyperbola and to the straight line being

$$r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1,$$

and
$$\delta = r\cos(\theta - \lambda),$$

we have, at their intersections,

$$\delta^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = \cos^2(\theta - \lambda),$$

and therefore

$$\left(\sin^2\lambda + \frac{\delta^2}{b^2}\right)\tan^2\theta + \sin 2\lambda . \tan\theta + \cos^2\lambda - \frac{\delta^2}{a^2} = 0,$$

and consequently, $\theta_{,}$, $\theta_{,,}$, being the inclinations of CP, CP', to the transverse axis,

$$\tan(\theta_{,} + \theta_{,,}) = \frac{\sin 2\lambda}{\cos 2\lambda - \delta^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}.$$

COR. Suppose that $\frac{b}{e} = c$, e denoting the eccentricity of the hyperbola: then we shall have

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2},$$

and therefore $$\tan(\theta_{,} + \theta_{,,}) = \frac{\sin 2\lambda}{\cos 2\lambda - \frac{\delta^2}{c^2}}.$$

This result shews that the angle $\theta_{,} + \theta_{,,}$ remains invariable for all hyperbolas with axes coincident in direction, provided that their *minor directors** are coincident.

Booth: *Application of a New Analytic Method to the Theory of Curves and Curved Surfaces*, p. 7.

3. To prove that diameters of an hyperbola which are nearer the transverse axis are smaller than those which are more remote.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 1.

4. The distances r, s, of two points in an hyperbola and its conjugate respectively from the common centre are at right angles to each other: to prove that, a, b, being the semi-axes of the hyperbola,

$$\frac{1}{r^2} - \frac{1}{s^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

SECTION XV.

Polar Equation. Focus the Pole.

1. If a straight line be drawn from the focus of an hyperbola, to prove that the part intercepted between the curve and the asymptote is equal to

$$\frac{a \sin\alpha}{\sin\alpha + \sin\theta},$$

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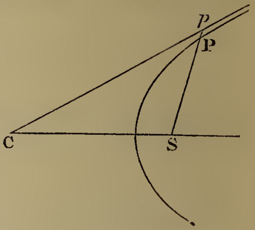
* A line drawn parallel to the transverse axis at a distance $\frac{b}{e}$ from the centre has been called the *minor director*.

where θ and α are the angles made by the straight line and asymptote respectively with the axis of the curve.



Let p, P, be the points in which the straight line, through the focus S, cuts the asymptote and curve respectively.

Then $\tan^2\alpha = \frac{b^2}{a^2} = e^2 - 1,$

$e^2 = \sec^2\alpha, \quad e = \sec\alpha.$

$$Sp = ae\frac{\sin\alpha}{\sin(\theta-\alpha)},$$

$$SP = \frac{a(e^2-1)}{1-e\cos\theta}.$$

Hence
$$Pp = \frac{ae\sin\alpha}{\sin(\theta-\alpha)} - \frac{a(e^2-1)}{1-e\cos\theta}$$

$$= a\tan\alpha\left\{\frac{1}{\sin(\theta-\alpha)} - \frac{\sin\alpha}{\cos\alpha-\cos\theta}\right\}$$

$$= a\tan\alpha\left\{\frac{1}{\sin(\theta-\alpha)} - \frac{\sin\alpha(\cos\alpha+\cos\theta)}{\sin(\theta+\alpha)\sin(\theta-\alpha)}\right\}$$

$$= \frac{a\tan\alpha}{\sin(\theta-\alpha)}\cdot\frac{\sin\theta\cos\alpha-\sin\alpha\cos\alpha}{\sin(\theta+\alpha)}$$

$$= 2a\sin\alpha\frac{\sin\theta-\sin\alpha}{\cos 2\alpha-\cos 2\theta}$$

$$= a\sin\alpha\frac{\sin\theta-\sin\alpha}{\sin^2\theta-\sin^2\alpha} = \frac{a\sin\alpha}{\sin\alpha+\sin\theta}.$$

2. To prove that $r\cos\frac{3\theta}{2} = a\cos\frac{\theta}{2}$ is a focal polar equation to an hyperbola, the eccentricity of which is 2 and the axes $\frac{2a}{3}, \frac{2a}{\sqrt{3}}$.

SECTION XVI.

Polar Equation. Point in the Axis the Pole.

1. In the semi-axis major CA of an hyperbola, produced indefinitely, a point O is taken at a distance from A greater than half the latus-rectum. P is a point in the curve, OP is joined. To determine the least value of OP.

Let $OP = r$, $\angle POA = \theta$, $CO = c$. Then the polar equation to the curve will be

$$r^2 \sin^2\theta = (e^2 - 1).\{(c - r\cos\theta)^2 - a^2\},$$

or $$e^2r^2\cos^2\theta - 2(e^2 - 1)\,cr\cos\theta = r^2 - (e^2 - 1)(c^2 - a^2).$$

This equation may be put under the form

$$\left(er\cos\theta - \frac{e^2 - 1}{e}c\right)^2 = r^2 - \frac{e^2 - 1}{e^2}.(c^2 - e^2a^2).....(1).$$

Now, since AO is greater than $a(e^2 - 1)$, c is greater than e^2a, and therefore à fortiori than ea: hence,

$$\frac{e^2 - 1}{e^2}(c^2 - e^2a^2) = \text{an essentially positive quantity.}$$

Hence it follows that

$$\frac{e^2 - 1}{e^2}(c^2 - e^2a^2)$$

is the least value of r^2.

From (1) we have, for the determination of the corresponding value of θ,

$$\cos\theta = \frac{c}{e}.\left(\frac{e^2 - 1}{c^2 - e^2a^2}\right)^{\frac{1}{2}},$$

which, c being greater than e^2a, gives possible values for θ.

Apollonius: *Conicorum*, lib. v. prop. 9.

2. A point O is taken in the semi-axis major (produced) of an hyperbola, at a distance from its vertex A equal to half the latus-rectum. A point P is taken in the curve, PO is joined, and PM is drawn at right angles to AO. To prove that OA is the least value of OP, and that, e denoting the eccentricity, $(OP)^2 - (OA)^2 = e^2(AM)^2$.

Apollonius: *Conicorum*, lib. v. prop. 5.

SECTION XVII.

Polar Equation. Pole Anywhere.

1. Through any given point O to draw a straight line cutting an hyperbola in two points Q, q, such that the rectangle contained by OQ, Oq, shall be equal to a given square.

The equation to the hyperbola, referred to its principal axes, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Let (h, k), be the coordinates of O. Then the polar equation to the hyperbola, O being the pole, and the prime radius vector being parallel to the axis of x, will be

$$\frac{(r\cos\theta + h)^2}{a^2} - \frac{(r\sin\theta + k)^2}{b^2} = 1.$$

Let r_1, r_2, be the two values of r in this quadratic: then

$$r_1.r_2 = \frac{\dfrac{h^2}{a^2} - \dfrac{k^2}{b^2} - 1}{\dfrac{\cos^2\theta}{a^2} - \dfrac{\sin^2\theta}{b^2}}.$$

Let c^2 denote the area of the given square: then $r_1.r_2 = \pm c^2$, the $-$ or $+$ sign being taken accordingly as O is supposed to lie between Q, q, or not: hence

$$\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \pm\frac{1}{c^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right).$$

From this equation may be found generally two values of θ, and therefore the problem will generally admit of two solutions on each hypothesis respecting the position of O. The possibility of the problem depends upon the possibility of the value of θ.

De la Hire: *Sectiones Conicæ*, lib. V. prop. 43.

2. Through a given point O is drawn a straight line cutting in the points P, Q, the opposite branches of an hyperbola: to find the inclination of PQ to the transverse axis in order that the rectangle between OP, OQ, may be the greatest possible.

PQ must be parallel to the transverse axis.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 32.

3. Through a given point O is drawn a straight line cutting in the points P, Q, one branch of an hyperbola: to prove that the rectangle between OP, OQ, will be the least possible when PQ is parallel to the conjugate axis.

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 34.

SECTION XVIII.

Poles and Polars.

1. With the asymptotes of an hyperbola, as conjugate diameters, ellipses are described, touching the hyperbola: to prove that, if a common polar of any two of the series of ellipses has its pole, relatively to one of the two ellipses, in the hyperbola, its pole, relatively to the other, will also be in the hyperbola.

Let the equations to any pair of the ellipses be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad\text{..........................}\quad (1),$$

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1 \quad\text{..........................}\quad (2).$$

Let the pole, relatively to (1), be (h, k): then the polar will be

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad\text{..........................}\quad (3).$$

Let (h', k') be the pole, relatively to (2), of the polar (3). Then the equation to the polar must be

$$\frac{h'x}{a'^2} + \frac{k'y}{b'^2} = 1 \quad\text{..........................}\quad (4).$$

Since (3) and (4) represent the same line,

$$\frac{h}{a^2} = \frac{h'}{a'^2}, \qquad \frac{k}{b^2} = \frac{k'}{b'^2},$$

and therefore

$$\frac{hk}{a^2b^2} = \frac{h'k'}{a'^2b'^2} \quad\text{..........................}\quad (5).$$

Since (1) and (2) touch the hyperbola, the equation of which is of the form

$$xy = c^2,$$

it may easily be shewn that $a^2b^2 = 4c^2$, $a'^2b'^2 = 4c^2$. Hence, by (5),

$$hk = h'k'.$$

But $hk = c^2$, because (h, k) is a point in the curve: consequently $h'k' = c^2$, and (h', k') is also a point in the curve.

2. If P be the pole, relatively to any polar of an equilateral hyperbola, of which C is the centre, to prove that ϕ, ϕ', being the inclinations of CP, and of the polar, to the real axis,

$$\tan\phi . \tan\phi' = 1.$$

Let the equation to the hyperbola be

$$x^2 - y^2 = a^2.$$

Then, (h, k) being the pole, the equation to the polar will be

$$hx - ky = a^2 \text{......................} (1).$$

Also the equation to CP will be

$$\frac{x}{h} = \frac{y}{k} \text{..........................} (2).$$

From (1) and (2) it appears that

$$\tan\phi = \frac{k}{h}, \quad \tan\phi' = \frac{h}{k};$$

and therefore $$\tan\phi . \tan\phi' = 1.$$

3. If a circle pass through the centre of an equilateral hyperbola and through any two points, A, B, to prove that it will also pass through the intersection of two lines, one drawn through A, parallel to the polar of B, and the other through B, parallel to the polar of A.

The equation to the hyperbola, referred to its asymptotes as axes, will be of the form

$$xy = c^2.$$

Let h, k, be the coordinates of A, and h', k', those of B. Then the equation to the polar of A will be

$$kx + hy = 2c^2 \text{......................} (1),$$

and that to the polar of B will be

$$k'x + h'y = 2c^2 \text{......................} (2).$$

The equation to the line through A, parallel to the line (2), will be

$$k'(x-h)+h'(y-k)=0\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

and that to the line through B, parallel to (1), will be

$$k(x-h')+h(y-k')=0\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation to any circle passing through the centre of the hyperbola will be

$$x(x+\alpha)+y(y+\beta)=0\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If this circle passes through the two points (h, k), (h', k'), we shall have

$$h(h+\alpha)+k(k+\beta)=0\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and

$$h'(h'+\alpha)+k'(k'+\beta)=0\ldots\ldots\ldots\ldots\ldots(7).$$

From (4) and (6), we have

$$(x-h')(h+\alpha)-(y-k')(k+\beta)=0,$$

and, from (3) and (7),

$$(x-h)(h'+\alpha)-(y-k)(k'+\beta)=0.$$

From the last two equations, we get, by subtraction,

$$(x+\alpha)(h-h')-(y+\beta)(k-k')=0\ldots\ldots\ldots(8).$$

But, from (3) and (4), by subtraction,

$$(k-k')x+(h-h')y=0\ldots\ldots\ldots\ldots\ldots(9).$$

From (8) and (9), there is

$$x(x+\alpha)+y(y+\beta)=0,$$

an equation identical with (5). Thus the intersection of (3) and (4) must lie in (5).

Brianchon, Poncelet: *Gergonne*, *Annales de Mathématiques*, tom. XI. p. 208.

4. When three points, situated in the plane of an equilateral hyperbola, are such that each of them is the pole of the line joining the other two, the circle which passes through these three points will pass also through the centre of the hyperbola.

Let the three points be (h, k), (h', k'), (h'', k''). The equation to the polar of the first point will be

$$kx+hy=2c^2.$$

Since this line contains the two other points, we have

$$kh'+hk'=2c^2,\quad kh''+hk''=2c^2,$$

and therefore

$$k(h''-h')+h(k''-k')=0\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly $$k'(h-h'')+h'(k-k'')=0 \quad\text{..............}(2),$$
and $$k''(h'-h)+h''(k'-k)=0 \quad\text{..............}(3).$$

Let the equation to the circle be

$$x^2+y^2+\alpha x+\beta y+\gamma=0.$$

Then, α, β, γ, will be subject to the relations

$$h^2+k^2+\alpha h+\beta k+\gamma=0 \quad\text{............}(4),$$
$$h'^2+k'^2+\alpha h'+\beta k'+\gamma=0 \quad\text{............}(5),$$
$$h''^2+k''^2+\alpha h''+\beta k''+\gamma=0 \quad\text{............}(6).$$

Subtracting (5) from (6), we get, by the aid of (1),

$$-h(h''+h')+k(k''+k')-\alpha h+\beta k=0 \quad\text{......}(7).$$

Similarly, from (6), (4), and (2), there is

$$-h'(h+h'')+k'(k+k'')-\alpha h'+\beta k'=0 \quad\text{......}(8).$$

Subtracting (8) from (7), we have

$$h''(h'-h)-k''(k'-k)+\alpha(h'-h)-\beta(k'-k)=0,$$

and therefore, by (3),

$$h''^2+k''^2+\alpha h''+\beta k''=0 \quad\text{..............}(9).$$

From (6) and (9) we see that $\gamma=0$. Hence the circle passes through the centre of the hyperbola.

Brianchon, Poncelet: *Gergonne, Annales de Mathématiques*, tom. XI. p. 210.

5. To prove that the diameter terminating at the point of contact of a tangent to an equilateral hyperbola, bisects two of the four angles, formed by the two diameters which pass through the points of intersection of this tangent with the polars of the two foci.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XIX. p. 351.

6. To prove that the diameters terminating at the two extremities of a chord of an equilateral hyperbola, form respectively equal angles with those which pass through the intersections of this chord with the polars of the two foci.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XIX. p. 351.

SECTION XIX.

Hyperbolic Loci.

1. Each of a series of parabolas is described so as to pass through two given points, and have its axis parallel to a given fixed line: to find the locus of their foci.

Let r, r', be the distances of the two given points from the focus or directrix of any one of the parabolas, and h, h', their distances from a fixed line drawn at right angles to their axes. Then, the directrix of each parabola, under the condition of the problem, being parallel to this fixed line, it is plain that

$$r' - r = h' - h = c,$$

where c denotes some constant quantity.

The locus of the focus is therefore an hyperbola of which the two fixed points are the foci.

2. From a given point (a, b) is drawn a normal to the hyperbola $xy = c^2$, of which the asymptotes are perpendicular to each other. To find the locus of the intersection of the normal with the curve, when the parameter c is regarded as variable.

The equation to a normal at a point (x, y) of the curve is

$$x(x' - x) = y(y' - y).$$

Let this normal contain the point (a, b): then

$$x(a - x) = y(b - y),$$

the equation to the required locus, which is therefore another rectangular hyperbola.

COR. If the origin be removed to a point $(\frac{1}{2}a, \frac{1}{2}b)$, the equation becomes

$$x^2 - y^2 = \tfrac{1}{4}(a^2 - b^2).$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 263.

3. If tangents are drawn making given angles with the axes of all ellipses having the same given foci: to find the curve which is the locus of the points of contact.

Let the equation to one of the ellipses be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ .... } (1).$$

The tangent at any point (x, y) will be

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

and, δ being the distance of the origin from this tangent and α the inclination to the axis of x, we have

$$\frac{x}{a^2} = \frac{\cos\alpha}{\delta}, \qquad \frac{y}{b^2} = \frac{\sin\alpha}{\delta},$$

whence $$\frac{x\sin\alpha}{a^2} = \frac{y\cos\alpha}{b^2} \text{ .... } (2).$$

But, c being a constant quantity,

$$a^2 - b^2 = c^2 \text{ .... } (3).$$

From (1) and (3),

$$\frac{x^2}{a^2}\cdot(a^2 - b^2) + \frac{y^2}{b^2}\cdot(a^2 - b^2) = c^2,$$

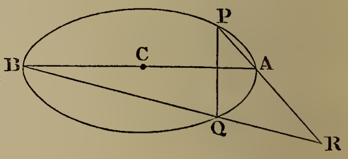
and therefore, from (2),

$$(x\cos\alpha + y\sin\alpha)(x\sin\alpha - y\cos\alpha) = c^2 \sin\alpha \cos\alpha,$$

or $$(x^2 - y^2)\sin 2\alpha - 2xy\cos 2\alpha = c^2 \sin 2\alpha,$$

which is the equation to an hyperbola concentric with the series of ellipses.

4. *PQ* is a chord of an ellipse at right angles to the axis major *AB*. The two lines *AP*, *BQ*, produced indefinitely, intersect in *R*. To find the locus of *R*.



The axes of the ellipse being taken as axes of coordinates, the equation to *PR* will be

$$y = -(x - a)\cdot\frac{y'}{a - x'},$$

x', y', being the coordinates of *P*.

The equation to BR will be

$$y = -(x+a) . \frac{y'}{a+x'} .$$

At the intersection of these two lines, we shall have

$$y^2 = (x^2 - a^2) . \frac{y'^2}{a^2 - x'^2};$$

or, by virtue of the equation to the ellipse,

$$y^2 = \frac{b^2}{a^2} (x^2 - a^2).$$

The required locus is therefore an hyperbola the axes of which coincide with those of the ellipse.

5. In two hyperbolas, concentric and similarly situated, are taken any two points the abscissæ of which are as the real axes of the curves: to find the locus of the middle point of the line joining the two points.

Let (x_1, y_1) be the point in one hyperbola, and (x_2, y_2) that in the other. Then, (a, b) and (α, β) being the semi-axes in the two curves,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{x_2^2}{\alpha^2} - \frac{y_2^2}{\beta^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, by the hypothesis,

$$\frac{x_1}{a} = \frac{x_2}{\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1), (2), (3), we see that

$$\frac{y_1}{b} = \pm \frac{y_2}{\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let x, y, be the coordinates of the middle point between (x_1, y_1) and (x_2, y_2): then

$$2x = x_1 + x_2,$$

$$2y = y_1 + y_2,$$

and therefore, by (3) and (4),

$$\frac{x_1}{a} = \frac{2x}{a+\alpha},$$

$$\frac{y_1}{b} = \frac{2y}{b \pm \beta}.$$

Substituting these expressions for $\frac{x_1}{a}$ and $\frac{y_1}{b}$ in (1), we get

$$\frac{x^2}{(a+\alpha)^2} - \frac{y^2}{(b \pm \beta)^2} = \tfrac{1}{4},$$

the equation to the required locus, being therefore an hyperbola concentric with the original ones and similarly situated.

6. To find the locus of T, the intersection of the tangents at two points P, Q, of a parabola, such that, S being the focus, SP, SQ, include a given angle ε.

Let A be the vertex of the parabola, and $AS = m$. Then, if $\angle ASP = \alpha$, the equation to PT will be

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \alpha) \quad \text{..................(1).}$$

Similarly, if $\angle ASQ = \beta$, the equation to QT will be

$$\frac{2m}{r} = \cos\theta + \cos(\theta - \beta) \quad \text{..................(2).}$$

At the intersection of (1) and (2),

$$\cos(\theta - \alpha) = \cos(\theta - \beta),$$

$$\theta - \alpha = -(\theta - \beta),$$

$$\beta + \alpha = 2\theta \quad \text{..........................(3).}$$

But, by the hypothesis, $\beta - \alpha = \varepsilon$ ....(4).

From (3) and (4), $\theta - \alpha = \tfrac{1}{2}\varepsilon$,

and therefore, from (1), we get for the equation to the locus of T,

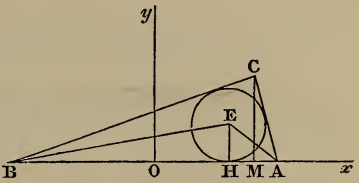
$$\frac{2m}{r} = \cos\theta + \cos\tfrac{1}{2}\varepsilon,$$

$$r = \frac{2m \sec\frac{1}{2}\varepsilon}{1 + \sec\frac{1}{2}\varepsilon . \cos\theta},$$

the equation to an hyperbola of which $\sec\frac{1}{2}\varepsilon$ is the eccentricity.

7. To find the locus of the vertex of a plane triangle, having given the radius of the inscribed circle, and the difference between the angles at the base; the middle point of the base being a fixed point, and the base itself lying along a fixed line.

Let O, the middle point of the base AB, be taken as the origin of coordinates, OA, produced indefinitely, as the axis of x, and Oy, at right angles to Ox, as the axis of y. Let E be the centre of the inscribed circle. Draw CM, EH, perpendicularly to AB.



Let $\angle CAB = 2\alpha$, $\angle CBA = 2\beta$, $AB = 2c$, $EH = r$, $OM = x$, $CM = y$, $\alpha - \beta = \varepsilon$.

Then it is clear from the geometry that

$$y = (c - x)\tan 2\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$y = (c + x)\tan 2\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$2c = r(\cot\alpha + \cot\beta) \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating c and x between (1), (2), (3), we have

$$y(\tan 2\alpha + \tan 2\beta) = r(\cot\alpha + \cot\beta)\tan 2\alpha.\tan 2\beta,$$

$$\begin{aligned} y\sin(2\alpha + 2\beta) &= r(\cot\alpha + \cot\beta).\sin 2\alpha.\sin 2\beta \\ &= 4r.\sin(\alpha + \beta).\cos\alpha.\cos\beta, \end{aligned}$$

$$\begin{aligned} y\cos(\alpha + \beta) &= 2r\cos\alpha\cos\beta \\ &= r\{\cos(\alpha + \beta) + \cos(\alpha - \beta)\}, \end{aligned}$$

$$\begin{aligned} (y - r)\cos(\alpha + \beta) &= r\cos(\alpha - \beta) \\ &= r\cos\varepsilon \ldots\ldots\ldots\ldots\ldots\ldots (4). \end{aligned}$$

Again, eliminating c between (1) and (2), we get

$$y(\tan 2\alpha - \tan 2\beta) = 2x\tan 2\alpha.\tan 2\beta,$$

$$\begin{aligned} y\sin(2\alpha - 2\beta) &= 2x\sin 2\alpha\sin 2\beta \\ &= x\{\cos(2\alpha - 2\beta) - \cos(2\alpha + 2\beta)\}, \end{aligned}$$

$$\begin{aligned} y\sin 2\varepsilon &= x\{\cos 2\varepsilon - 2\cos^2(\alpha + \beta) + 1\} \\ &= 2x\{\cos^2\varepsilon - \cos^2(\alpha + \beta)\}. \end{aligned}$$

Hence, by (4), $\quad y\sin 2\varepsilon = 2x\left\{\cos^2\varepsilon - \dfrac{r^2\cos^2\varepsilon}{(y - r)^2}\right\}$,

$$\begin{aligned} y\tan\varepsilon.(y - r)^2 &= x\{(y - r)^2 - r^2\} \\ &= xy(y - 2r), \end{aligned}$$

$$(y - r)^2.\tan\varepsilon = x(y - 2r),$$

the equation to the required locus, which is therefore an hyperbola.

Leybourn: *Mathematical Repository*, *New Series*, vol. II. p. 218.

8. From a given point E in the semi-axis major AC of an ellipse is drawn a straight line cutting the curve in any point P: PM is drawn to cut CA at right angles in M: MP is produced to a point P' such that $MP' = EP$. To find the locus of P'.

Let $CE = c$: then, the equation to the ellipse being

$$y^2 = (1 - e^2)(a^2 - x^2),$$

the locus of P' will be an hyperbola represented by the equation

$$e^2y^2 = (e^2x - c)^2 + (1 - e^2)(e^2a^2 - c^2).$$

P. Gregorius à Sancto Vincentio: *Opus Geometricum Quadraturæ Circuli et Sectionum Coni*, p. 659.

9. Two straight lines, of given lengths, coincide with and move along two fixed axes, in such a manner that a circle may always be drawn through their extremities: to find the locus of the centre of the circle.

Let a, b, be the lengths of the moveable lines on the axes of x, y, respectively, and let ω be the angle between the axes. Then the required locus will be an hyperbola denoted by the equation

$$x^2 - y^2 = \frac{a^2 - b^2}{4 \sin^2 \omega}.$$

10. Two given straight lines OA, OB, intersecting in O, are touched by a parabola in points H, K, respectively: the parallelogram $OHPK$ is completed. Supposing the parabola to touch also a third given line, to find the locus of P.

If the lines OA, OB, be taken as axes of coordinates, and the equation to the third given line be

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1,$$

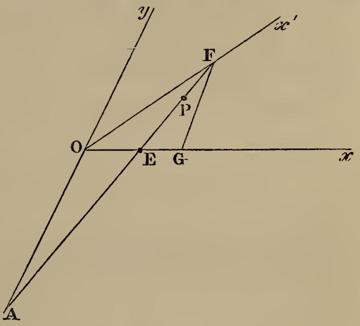
the equation to the required locus will be

$$\frac{\alpha}{x} + \frac{\beta}{y} = 1,$$

which is therefore an hyperbola, the asymptotes of which are represented by the equations

$$x = \alpha, \quad y = \beta.$$

11. Ox, Ox', are two indefinite straight lines, given in position: from a given point A is drawn a straight line AEF, cutting Ox, Ox', in E, F, respectively: EF is divided into two parts EP, FP, which bear to each other always a constant ratio. To find the locus of the point P.



Join AO and produce it indefinitely to y: let Ox, Oy, be chosen as axes of coordinates. Draw FG parallel to yO. Let $OA = a$; and let

$$EF : EP :: n : m,$$

and

$$OG : FG :: l : n.$$

Then the required locus will be an hyperbola represented by the equation

$$(ny + ma)\, x = ly\,(y + a).$$

Newton: *Arithmetica Universalis*, prob. 25.

12. The base OA of a triangle OAP is given: to find the locus of the vertex, the angle PAO being always double the angle POA.

Let OA produced indefinitely be taken as the axis of x, and a perpendicular to OA through O as that of y. Let $OA = c$. The required locus will therefore be an hyperbola, the asymptotes of which include an angle of 120°, and which is represented by the equation

$$y^2 = 3x^2 - 2cx.$$

Pappus: *Mathematicæ Collectiones à Commandino*, lib. IV. prop. 34.

Newton: *Arithmetica Universalis*, prob. 36.

13. To find the locus of the vertex of a triangle, the base of which is given, the two adjacent angles differing by a given quantity.

Let $c =$ the base AB of the triangle, the angle at B exceeding that at A by a quantity α: let AB, produced indefinitely, be the axis of x, a perpendicular to it through A being that of y. Then the locus required will be an hyperbola defined by the equation

$$(x^2 - y^2)\sin\alpha + 2xy\cos\alpha = c\,(x\sin\alpha + y\cos\alpha).$$

Puissant: *Recueil de diverses Propositions de Géométrie*, p. 204, *troisième édition.*

14. If normals be drawn to an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

from a given point (α, β) in its area, to prove that the points where they cut the curve will all lie in a rectangular hyperbola of which the equation is

$$(a^2 - b^2)\,xy = a^2\alpha y - b^2\beta x.$$

15. A circle always passes through the origin of coordinates and through the points in which the axes are intersected by a line

$$\frac{x}{m} + \frac{y}{n} = 1,$$

which always passes through a given point: to find the locus of the centre of the circle, m and n being variable, and the angle between the axes being given.

If ω denote the angle between the axes, and (α, β) be the given point, the locus will be an hyperbola represented by the equation

$$2\cos\omega\,(x^2+y^2)+2\,(1+\cos^2\omega)\,xy-(\alpha\cos\omega+\beta)\,x-(\beta\cos\omega+\alpha)\,y=0.$$

Leybourn's *Mathematical Repository*, No. XXIII. p. 74.

16. Two tangents to a parabola include a constant angle: to find the locus of their point of intersection.

The equation to the parabola being

$$y^2 = 4mx,$$

and ε denoting the constant angle, the required locus will be an hyperbola defined by the equation

$$y^2 = 4mx + \tan^2\varepsilon . (x + m)^2.$$

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 29.
L'Hospital: *Traité Analytique des Sections Coniques*, p. 266.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 210, *troisième édition.*

17. If any straight line, parallel to a given straight line, be drawn cutting an ellipse, and from the points of intersection normals be drawn, to find the locus of the intersection of these normals.

The equation to the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the direction-cosines of the given line being proportional to α, β, the required locus will be an arc of the hyperbola defined by the equation

$$\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) . \frac{a^2\alpha x + b^2\beta y}{(a^2 - b^2)^2} = \left(\frac{a^2\alpha^2 - b^2\beta^2}{a^2\alpha^2 + b^2\beta^2}\right)^2.$$

Goodwin: *Cambridge Mathematical Journal*, vol. IV. p. 190.

18. A series of circles pass through a given point O, have their centres in a line OA, and meet another line AB. From the point, in which one of the circles again meets the line OA, is drawn a straight line, parallel to AB, to intersect, in points P and Q, straight lines drawn parallel to OA through the points in which the circle is cut by AB. To find the loci of the intersections P and Q.

The line OA being taken as the axis of x, and O being the origin, let the equation to AB be

$$x = \alpha y + \beta.$$

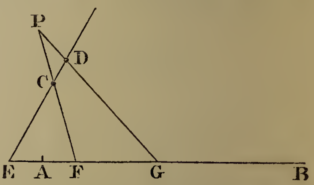
Then the points P and Q lie generally in an hyperbola, of which the equation is

$$(1 + 2\alpha^2)\, y^2 - \alpha xy + 3\alpha\beta y - \beta x + \beta^2 = 0.$$

If the lines OA, AB, be at right angles to each other, $\alpha = 0$, and the locus becomes a parabola.

19. A straight line DCE, passing through two given points D, C, cuts a given straight line BA, produced if necessary, in E. In AB are taken any two points F, G, such that

$$\frac{AF}{BG} = \frac{n}{n'},$$



n and n' being constants. The straight lines FC, GD, are produced to meet in P. To find the locus of P.

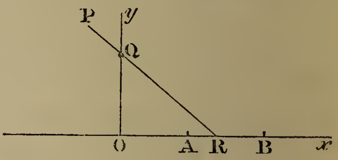
Let EB, ED, produced indefinitely, be taken as axes of x, y, respectively: let $EC = k$, $ED = k'$, $EA = a$, $EB = a'$. Then the required locus will be an hyperbola defined by the equation

$$x\left(\frac{n'k}{k-y} + \frac{nk'}{k'-y}\right) = n'a + na'.$$

Barrow: *Geometrical Lectures*, Lect. VI.
Leybourn: *Mathematical Repository*, *New Series*, vol. I. p. 45.

20. Let AB be a given straight line, and Q a given point without it: let PQR be drawn, from a point P, to meet AB in such a point R, that

$$\frac{RQ}{PQ} = \frac{AR}{BR}.$$



To find the locus of P.

Draw through Q the line QO at right angles to the direction of AB, and cutting it in O: let OB, OQ, produced indefinitely, be taken as axes of x, y, respectively. Let $OA = a$, $OB = a'$, $OQ = c$. Then the locus of P will be an hyperbola defined by the equation $cxy + (y-c)\,\{ca' + a\,(y-c)\} = 0.$

COR. It may be easily shewn that the line BQ is a tangent to the hyperbola at the point Q.

Leybourn: *Mathematical Repository*, *New Series*, vol. I. p. 146.

21. To find the locus of the centre of a circle which touches externally two given unequal circles.

The locus will be an hyperbola the two foci of which are the centres of the two given circles.

Puissant: *Recueil de diverses Propositions de Géométrie*, p. 207, *troisième édition.*

22. To find the locus of a point, within a given angle, such that, perpendiculars being drawn from it upon the sides of the angle, the quadrilateral so formed may be of a given area.

Let c^2 be the given area; ω the given angle, its sides being taken as axes of coordinates. Then the required locus will be an hyperbola the equation of which is

$$(x^2 + y^2) \cos \omega + 2xy = 2c^2 \operatorname{cosec} \omega.$$

Puissant: *Recueil de diverses Propositions de Géométrie*, p. 217, *troisième édition.*

23. Having given one side of a triangle, and the difference between the tangents of the adjacent angles, to find the locus of the vertex.

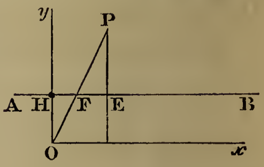
Take the middle point of the given side as the origin of rectangular coordinates, the side coinciding with the axis of x; then, m denoting the given difference, and $2a$ the given side, the required locus will be an hyperbola defined by the equation

$$2xy = m (a^2 - x^2).$$

Lardner: *Algebraic Geometry*, p. 117.

24. Having given in position a right line AB, and a point O outside of it, a right line OP is drawn cutting AB in F, and from P is drawn a perpendicular PE upon AB: the magnitude of EF being invariable, to find the locus of P.

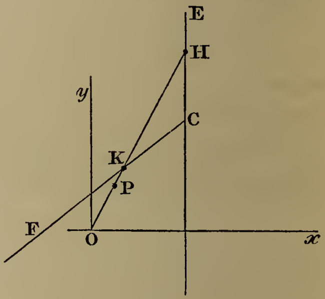
Take Ox, parallel to AB, as the axis of x, and Oy, at right angles to AB, as that of y. Let $EF = a$, $OH = b$. Then the locus of P is a rectangular hyperbola represented by the equation



$$xy = ay + bx.$$

Lardner: *Algebraic Geometry*, p. 134.

25. Through a given point O a right line HKO is drawn, intersecting two right lines EC, FC, in H, K: a part OP of OH being supposed to be always equal to HK, to find the locus of P.



Let Ox, at right angles to the line CE, be the axis of x, and Oy, parallel to CE, that of y. Let β denote the angle ECF. Then the required locus will be an hyperbola defined by the equation

$$xy + x^2 \cot\beta - ay + bx = 0,$$

where a, b, are the coordinates of C.

Lardner: *Algebraic Geometry*, p. 135.

26. From the point of intersection of two given straight lines, is drawn a straight line of given length, bisecting the angle between them: to determine the locus of the middle point of a straight line, drawn through the extremity of this line, and terminating in the other two.

Let the two given lines be taken as axes of coordinates, 2ω denoting the angle between them: let c represent the given length of the bisecting line. Then the required locus will be an hyperbola, defined by the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{4\cos\omega}{c}.$$

27. Two fixed axes Ox, Oy, intersect a variable circle in points (A, A'), (B, B'), respectively. To find the locus of P, the

intersection of the chords AB', $A'B$, supposing the two points B, B', to be fixed.

If $OB = b$, $OB' = b'$, the equation to the required locus is

$$x^2 - y^2 + (b + b')\,y = bb'.$$

28. A straight line BC moves between two fixed straight lines AB, AC, so as to make the area ABC constant: to find the locus of the centre of a circle described about the triangle ABC.

Let AB, AC, produced indefinitely, be taken as axes of x, y, respectively: let ω be the angle between the axes. Then, the area ABC being constant, the rectangle between AB, AC, will be constant: denoting this latter area by $4m^2$, we shall have for the equation to the required locus

$$(x + y \cos\omega) \,.\, (y + x \cos\omega) = m^2.$$

29. A and B are two given points, Ax an indefinite fixed line: through B is drawn a straight line cutting Ax in T, and through A is drawn a straight line AP cutting BT in P. To find the locus of P, having given that AP is equal to PT.

Let Ax be taken as the axis of x, and a line through A, at right angles to it, as that of y. Let $2a$, $2b$, be the coordinates of B. Then the required locus will be an hyperbola represented by the equation

$$(x - a)\,y = bx.$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 271.

SECTION XX.

Miscellaneous Problems.

1. From a given point S, in the plane of a given circle, a straight line is drawn to intersect the circumference in P: from the point P an indefinite straight line PQ is drawn perpendicularly to SP: to prove that PQ is always a tangent to an ellipse or hyperbola the centre of which coincides with the centre C of the circle and of which S is a focus.

Let C be taken as the origin of coordinates; CS, produced indefinitely, as the axis of x. Let $SP = m$, $CS = ae$, a being the radius of the circle, and $\angle PSx = \theta$.

Then, x, y, being the coordinates of P,

$$m \cos\theta + ae = x, \quad m \sin\theta = y,$$

and therefore $$m^2 + 2aem \cos\theta = a^2(1 - e^2) \ldots\ldots\ldots\ldots\ldots (1).$$

Also, the equation to PQ will be

$$x \cos\theta + y \sin\theta = ae \cos\theta + m \ldots\ldots\ldots\ldots (2).$$

By virtue of (1), (2) becomes

$$(x \cos\theta + y \sin\theta)^2 = a^2 \cos^2\theta + a^2(1 - e^2) \sin^2\theta.$$

This equation shews that PQ is the tangent to an ellipse or hyperbola of which C is the centre and S a focus.

This proposition affords a convenient method of drawing a tangent to an ellipse or hyperbola, which shall be parallel to a given line. From either focus S draw a line SP, at right angles to the given line, to intersect in P the circle described on the axis major. From P draw PQ perpendicularly to SP. Then PQ will be the required tangent.

Maclaurin: *Geometria Organica*, sect. III. p. 102.
Prony: *Journal de l'Ecole Polytechnique*, cahier X. p. 49.
Gergonne: *Annales de Mathématiques*, tom. V. p. 49.

2. In every line of the second order, which has a centre, if we draw two tangents parallel to any single fixed line, and a third variable tangent; the product of the segments of the first two tangents comprised between their points of contact and the third tangent, will be a constant quantity.

The points of contact of the two parallel tangents being the extremities of a diameter, we will take this diameter, which we will call $2a$, for the axis of x, and its conjugate $2b$ for the axis of y.

If x', y', be the coordinates of the point of contact of the third tangent, we shall have

$$\frac{x'^2}{a^2} \pm \frac{y'^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and the equation to this tangent will be

$$\frac{xx'}{a^2} \pm \frac{yy'}{b^2} = 1 \text{ ....} (2).$$

Let $y_{,}$, $y_{,,}$, be the values of y, given by (2), when x is successively equated to a and $-a$. The quantities $y_{,}$, $y_{,,}$, are the segments of the two parallel tangents. Then we have

$$y_{,} = \pm \frac{b^2}{ay'}(a - x'), \quad y_{,,} = \pm \frac{b^2(a + x')}{ay'};$$

and therefore

$$y_{,}y_{,,} = \frac{b^4}{a^2y'^2}(a^2 - x'^2)$$

$$= \pm b^2, \text{ by (1).}$$

Bérard: *Gergonne, Annales de Mathématiques*, tom. v. p. 52.
Brianchon: *Ibid.* p. 53.

3. A straight line, remaining always parallel to itself, moves in the plane of two fixed straight lines. In every position of the moveable line, a point is taken in it such that the sum or the difference of the squares of its distances from the intersections of the line with the two fixed lines is equal to a given square; to determine the locus of the point.

Let the two fixed straight lines be chosen as the axes of coordinates.

Let x, y, be the coordinates of the particular point in the moveable line, and (x', y') of any other point in the line. Then, r being the distance between these two points, the equations to the line will be

$$x' = lr + x,$$

$$y' = mr + y,$$

l and m being constants.

Let $r_{,}$, $r_{,,}$, be the values of r at the intersections of the moveable line with the axes of x, y, respectively: then

$$0 = lr_{,,} + x, \quad 0 = mr_{,} + y,$$

and therefore

$$r_{,}^2 \pm r_{,,}^2 = \frac{x^2}{l^2} \pm \frac{y^2}{m^2}.$$

But $r_{,}^2 \pm r_{,,}^2$ is equal to some constant c^2: hence the equation to the required locus is

$$\frac{x^2}{l^2} \pm \frac{y^2}{m^2} = c^2,$$

which belongs to an ellipse or an hyperbola.

Bret: *Gergonne, Annales de Mathématiques*, tom. VI. p. 12.

4. The equation to an hyperbola, referred to rectangular axes, being
$$ax^2 + by^2 + 2cxy = c',$$
to find the position of its asymptotes, and the equation to the hyperbola referred to them as axes.

But $x = r\cos\theta$, $y = r\sin\theta$: then

$$r^2(a\cos^2\theta + b\sin^2\theta + 2c\cos\theta\,.\sin\theta) = c'.$$

Now $r = \infty$, provided that

$$a\cos^2\theta + b\sin^2\theta + 2c\cos\theta\,.\sin\theta = 0,$$

and thus the equation to the two asymptotes is

$$ax^2 + by^2 + 2cxy = 0.$$

Again, a_1, $\dfrac{b_1}{\sqrt{(-1)}}$, representing the transverse and conjugate axes respectively, the equation to the hyperbola referred to its asymptotes is
$$xy = \tfrac{1}{4}(a_1^2 - b_1^2).$$
Differentiating
$$r^2 = x^2 + y^2,$$
we have, for maximum values of r,

$$xdx + ydy = 0,$$

and, from the equation to the curve,

$$axdx + bydy + c(xdy + ydx) = 0;$$

and therefore, λ being arbitrary,

$$\lambda x = ax + cy \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\lambda y = by + cx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Multiplying (1) by x, (2) by y, adding, and attending to the equation to the curve, we have

$$\lambda r^2 = c'.$$

But, from (1) and (2),

$$(\lambda - a)\, x = cy, \quad (\lambda - b)\, y = cx,$$

and therefore $$(\lambda - a)(\lambda - b) = c^2;$$

hence $$(c' - ar^2)(c' - br^2) = c^2 r^4,$$

$$(c^2 - ab)\, r^4 + c'(a + b)\, r^2 - c'^2 = 0.$$

Consequently, a_1^2, b_1^2, being the roots of this equation,

$$a_1^2 + b_1^2 = -c'\frac{a+b}{c^2 - ab}, \quad a_1^2 b_1^2 = -\frac{c'^2}{c^2 - ab},$$

and therefore $$(a_1^2 - b_1^2)^2 = c'^2 . \frac{(a-b)^2 + 4c^2}{(c^2 - ab)^2}.$$

Thus the equation to the hyperbola referred to its asymptotes is

$$xy = \tfrac{1}{4} c' . \frac{\{(a-b)^2 + 4c^2\}^{\frac{1}{2}}}{c^2 - ab}.$$

5. CP is any semi-diameter of an hyperbola, CD being its conjugate semi-diameter: PK is a tangent at P, equal to CD: from any point V in CP produced is drawn a straight line VEF, parallel to CD, cutting the curve in E and the straight line through C and K in F: to prove that

$$(VF)^2 - (VE)^2 = (CD)^2.$$

Pascal: *Essais pour les Coniques: Œuvres*, tom. IV. p. 5.

6. To prove that the radius of a circle, which touches an hyperbola and its asymptotes, is equal to that part of the latus-rectum produced which is intercepted between the curve and an asymptote.

7. If an ellipse and hyperbola have the same foci, to find the equation to the locus of the intersection of two tangents to them at right angles to one another.

The equations to the ellipse and hyperbola being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{and } \frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1,$$

the required locus will be a circle defined by the equation

$$x^2 + y^2 = a_1^2 + b^2.$$

8. P and Q being two points in an ellipse, if an hyperbola, the foci of which are P and Q, pass through one focus of the ellipse, to prove that it will pass also through the other focus.

9. To find the equations to the asymptotes of the hyperbola represented by the equation

$$xy + ax + by + c^2 = 0.$$

The required equations are

$$x + b = 0, \quad y + a = 0.$$

10. The equation to an hyperbola being

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

to shew that $$ax^2 + by^2 + 2cxy = 0$$

is the equation to two straight lines through the origin parallel to its asymptotes.

11. Three hyperbolas have parallel asymptotes: to shew that the three straight lines joining the points of intersection of the hyperbolas, taken two and two, all meet in one point.

12. To prove that an ellipse and hyperbola, which have the same centre and foci, will cut each other at right angles; and that, if from any point in the circumference of the circle, which passes through their points of intersection, tangents be drawn to the two curves, they will be at right angles to each other.

13. $CPLD$ is a parallelogram, the sides CP, CD, of which are conjugate semi-diameters of a rectangular hyperbola, inclined to one another at an angle of 60°: to find the equation to the ellipse which passes through the points C, P, L, D, and cuts the conjugate hyperbola at D at an angle of 15°.

The equation to the rectangular hyperbola being $x^2 - y^2 = a^2$, that to the required ellipse will be

$$\tfrac{1}{3}x^2 + y^2 + 2(2 - \sqrt{3})\, xy = \frac{a^2}{2\sqrt{3}}.$$

14. Any number of ellipses (or hyperbolas) concentric, similar, and similarly situated, are intersected by a line parallel to a directrix in P, P', P'',......: to prove that the extremities of the diameters respectively conjugate to the diameters through P, P', P'',...... are in a line perpendicular to the directrix.

15. If, in the preceding question, the curves be cut by any concentric hyperbola, the asymptotes of which have the same direction as their axes, in Q, Q', Q''......; to prove that the extremities of the diameters respectively conjugate to the diameters through Q, Q', Q''..., are situated in another branch of the hyperbola.

16. A chord PP' is drawn in an ellipse, so that, AC being the semi-axis major,

$$\tan PCA + \tan P'CA = 2 \tan\alpha ;$$

to find the envelop of the chord.

The required envelop is an hyperbola the equation of which is, the axes of coordinates coinciding with the axes of the ellipse,

$$xy - x^2 \tan\alpha = \frac{b^2}{4 \tan\alpha}.$$

17. To prove that the tangent to the interior of two similar hyperbolas with coincident axes, cuts off from the exterior curve a constant area.

De la Hire: *Sectiones Conicæ*, lib. VI. prop. 17.

18. Supposing that, y', y'', being any two values of y in the hyperbola $xy = c^2$, such that $y'' = \lambda y'$, where λ is constant, to prove that the area included between y', y'', the axis of x, and the curve, is invariable in magnitude.

Gregorius à Sancto Vincentio: *Opus Geometricum Posthumum ad Mesolabium*, p. 252.

LINES OF THE SECOND ORDER.

SECTION I.

Referred to a Principal Diameter and its Tangent. Normals.

1. FROM a point P of a conic section a normal is drawn to cut a principal axis (the major axis in the ellipse, the transverse axis in the hyperbola) in a point G: to prove that, of all straight lines drawn from G to the curve, GP is the least.

The equation to the conic section is

$$y^2 = (1+e)\,\{2emx - (1-e)\,x^2\}.$$

The equation to the normal, at any point (h, k), is

$$(1+e)\,\{em - (1-e)\,h\}\,.\,(y' - k) = -\,k\,(x' - h).$$

At the point G, $y' = 0$, and therefore

$$x' = (1+e)\,em + e^2h.$$

Let r represent the distance of G from any point (x, y) in the curve: then

$$\begin{aligned} r^2 &= \{x - (1+e)\,em - e^2h\}^2 + y^2 \\ &= x^2 - 2ex\,\{(1+e)\,m + eh\} + e^2\,\{(1+e)\,m + eh\}^2 \\ &\qquad + (1+e)\,\{2emx - (1-e)\,x^2\} \\ &= e^2\,(x-h)^2 + e^2\,(1+e)^2\,m^2 + 2e^3\,(1+e)\,mh - e^2\,(1-e^2)\,h^2, \end{aligned}$$

$$\begin{aligned} \frac{r^2}{e^2} &= (x-h)^2 + (1+e)^2\,m^2 + (1+e)\,\{2emh - (1-e)\,h^2\} \\ &= (x-h)^2 + (1+e)^2\,m^2 + k^2. \end{aligned}$$

Hence it appears that r is least when $x = h$, or when (x, y) coincides with (h, k).

De la Hire: *Sectiones Conicæ*, lib. VII. prop. 13.

2. *AP* is the arc of a conic section, of which the vertex is *A*; *PG* the normal, and *PK* a perpendicular to the chord *AP*, meet the axis in *G* and *K*. To shew that *GK* is equal to half the latus-rectum.

3. To find the locus of the middle point of the portion of the normal to a conic section which is included between the curve and the axis.

The equation to the conic section being

$$y^2 = mx + nx^2,$$

that to the required locus will be

$$(n+2)^2 y^2 - nx^2 - mx + \tfrac{1}{4}m^2. (n+1) = 0.$$

Lardner: *Algebraic Geometry*, p. 149.

SECTION II.

Referred to a Principal Diameter and its Tangent. Chords.

1. From one extremity *A* of a principal axis of a conic section, a given straight line *AP* is drawn to cut the curve in *P*; to find the equation to the line joining *P* to the other extremity *B* of the axis.

Let the axis *AB* and the tangent at *A* be taken as axes of x and y respectively. Then the equation to *AP* will be of the form

$$y = ax \text{ .... } (1),$$

and the equation to the conic section of the form

$$y^2 = mx + nx^2 \text{ .... } (2).$$

The equation to *AB* is $y = 0$ .... (3).

The two lines (1) and (3) may be represented simultaneously by the equation

$$y(y - ax) = 0,$$

or

$$y^2 = axy \text{ .... } (4).$$

Hence, at *A*, *P*, *B*, the intersections of *AP*, *AB*, with the curve,

$$axy = mx + nx^2,$$

whence dividing by x, we obtain for the equation to *BP*, which passes through *P*, *B*,

$$ay = m + nx.$$

2. If A and B be the extremities of the axis major of a conic section, T the point where a tangent at a point P in the curve, meets AB, QTR a line perpendicular to AB and meeting AP, BP, in Q, R, respectively; to prove that

$$QT = RT.$$

3. To find the length of the chord of a conic section, denoted by the equation

$$y^2 = 2mx + nx^2,$$

the equation to the chord being

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1.$$

If $2c$ denote the length of the chord,

$$c^2 = \frac{\alpha (\alpha^2 + \beta^2).(m^2\alpha + 2m\beta^2 + n\alpha\beta^2)}{(n\alpha^2 - \beta^2)^2}.$$

SECTION III.

Referred to a Principal Diameter and its Tangent. Focal Properties.

1. From the extremity L of the semi-latus-rectum SL of a conic section, a chord LA is drawn to the vertex A of the diameter through S. A tangent is drawn at L. A straight line MRP is drawn, through any point M in AS, or AS produced, at right angles to AS, meeting the chord AL in R, and the tangent at L in P. To prove that PR is equal to MS.

Taking AS, produced indefinitely, as the axis of x, and the tangent at A as that of y, the equation to the curve will be,

$$y^2 = (1 + e)\{2emx - (1 - e)\, x^2\}.$$

The equation to the tangent at any point (x, y) is

$$\frac{yy'}{1 + e} = \{em - (1 - e)\, x\}\, x' + emx.$$

Now the coordinates of L are em, $em(1+e)$: hence the equation to PL is

$$y' = e(x' + m) \text{..........................} (1).$$

Again, the equation to AL is

$$y' = (1+e)x' \text{..........................} (2).$$

Hence PR, which is the difference between the ordinates in (1) and (2), is equal to $em - x'$, that is, to MS.

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 14.

2. From the extremity L of the semi-latus-rectum SL of a conic section, a tangent LP is drawn. At right angles to the diameter through S, from any point M in it, a straight line is drawn to meet the tangent LP in P and the curve in Q. The points S, Q, are joined. To prove that $SQ = PM$.

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 15.

3. Having given the position of the focus S of a conic section, the position and magnitude of the semi-latus-rectum SL, and the position of T, the intersection of the tangent LT with SA produced, A being one end of the diameter through the foci: to determine the position of A, and of the other extremity A' of the axis AA'.

Let $ST = h$, $SL = k$: then

$$SA = \frac{hk}{h+k}, \qquad SA' = \pm \frac{hk}{h-k}.$$

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 21.

4. To prove that a straight line drawn through the focus of a conic section, to the point in which a diameter meets the directrix, will be perpendicular to a tangent at either extremity of that diameter.

Leybourn: *Mathematical Repository*, *New Series*, vol. II. p. 95.

SECTION IV.

Referred to any two Oblique Diameters.

1. CA, CB, are any two semi-diameters of a conic section. The tangent at A meets CB produced in Γ, and the tangent at B meets CA produced in Δ. To prove that, E being the intersection of $A\Gamma$, $B\Delta$, the triangle $A\Delta E$ is equal to the triangle $B\Gamma E$.

Let $CA = \alpha$, $CB = \beta$; then, CA, CB, produced indefinitely, being taken as axes of x, y, respectively, the equation to the curve will be

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + 2\frac{xy}{\gamma^2} = 1,$$

γ being some constant.

The equation to the tangent at any point (x, y) of the curve is

$$\left(\frac{x}{\alpha^2} + \frac{y}{\gamma^2}\right) x' + \left(\frac{y}{\beta^2} + \frac{x}{\gamma^2}\right) y' = 1.$$

Hence the equation to the tangent $A\Gamma$, drawn at the point $(\alpha, 0)$, is

$$x' + \frac{\alpha^2 y'}{\gamma^2} = \alpha.$$

Putting $x' = 0$, we see that

$$C\Gamma = y' = \frac{\gamma^2}{\alpha},$$

and therefore $$C\Gamma . CA = \gamma^2.$$

Similarly $$C\Delta . CB = \gamma^2.$$

Hence the triangles $AC\Gamma$, $BC\Delta$, and therefore the triangles $A\Delta E$, $B\Gamma E$, are equal.

Ἐὰν κώνου τομῆς ἢ κύκλου περιφερείας εὐθεῖαι ἐπιψαύουσαι συμπίπτωσιν, ἀχθῶσι δὲ διὰ τῶν ἁφῶν διάμετροι συμπίπτουσαι ταῖς ἐφαπτομέναις, ἴσα ἔσται τὰ γινόμενα κατὰ κορυφὴν τρίγωνα.

’ΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ *Κωνικῶν τὸ τρίτον· Πρότασις ά.*

2. To determine the magnitudes and positions of the axes of the conic section

$$ax^2 + by^2 + 2cxy = 1,$$

ω being the angle of coordination.

The magnitudes of the semi-axes are the positive values of r in the equation

$$(ab - c^2)\, r^4 - (a + b + 2c \cos\omega)\, r^2 + \sin^2\omega = 0;$$

and

$$(by - cx)(x - y\cos\omega) = (ax - cy)(y - x\cos\omega)$$

is the equation to the two axes.

O'Brien: *Cambridge Mathematical Journal*, vol. IV. p. 101.

SECTION V.

Referred to two Tangents as Axes.

1. If two conic sections have two points of contact, to prove that they will not meet in any other point.

Let the tangents at their points of contact be taken as axes of coordinates. Then, a, b, being the distances of the points of contact from the origin, the equations to the two conic sections will be

$$\frac{x^2}{a^2} + \frac{2xy}{mn} + \frac{y^2}{b^2} - 2\frac{x}{a} - 2\frac{y}{b} + 1 = 0,$$

$$\frac{x^2}{a^2} + \frac{2xy}{m_, n_,} + \frac{y^2}{b^2} - 2\frac{x}{a} - 2\frac{y}{b} + 1 = 0.$$

If the two curves meet, we have, subtracting one equation from the other,

$$xy = 0,$$

whence $x = 0$, and therefore $y = b$; or $y = 0$, and therefore $x = a$.

Thus we see that the two points of contact are the only points in which the two curves meet.

'Εὰν τῶν προειρημένων γραμμῶν τινες κατὰ δύο σημεῖα ἐφάπτωνται ἀλλήλων, οὐ συμβάλλουσιν ἀλλήλαις καθ' ἕτερον.

'ΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ *Κωνικῶν τὸ τέταρτον· Πρότασις κζ'.*

2. From a point O, two tangents OH, OK, are drawn to a conic section: a straight line MQR is drawn, parallel to OK,

to intersect OH in M, and the curve in Q, R. To prove that

$$(MH)^2 : MQ.MR :: (OH)^2 : (OK)^2.$$

Let OH, OK, produced indefinitely, be taken as axes of x, y, respectively. Let $OH = a$, $OK = b$. Then, m, n, being arbitrary constants, the equation to the curve will be

$$\frac{x^2}{a^2} + \frac{2xy}{mn} + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0.$$

Let $OM = h$: then, putting h for x in this equation, we shall have a quadratic in y, the two roots of which are MQ, MR.

Hence, equating the rectangle of these two lines to the last term of the quadratic, we have

$$MQ.MR = b^2 \left(\frac{h-a}{a}\right)^2$$

$$= \frac{(OK)^2.(MH)^2}{(OH)^2};$$

or $$(MH)^2 : MQ.MR :: (OH)^2 : (OK)^2.$$

De la Hire: *Sectiones Conicæ*, lib. III. prop. 26.

3. To find the locus of the centre of a conic section which touches two given straight lines at two given points.

The equation to the conic section, the two given straight lines being taken as axes of coordinates, will be of the form

$$a^2x^2 + 2cxy + b^2y^2 + 2ax + 2by + 1 = 0,$$

a and b being given constants.

If h, k, be the coordinates of the centre,

$$a^2h + ck + a = 0,$$

and $$b^2k + ch + b = 0.$$

From these two equations we have

$$(c^2 - a^2b^2)\,h = b\,(ab - c),$$

$$(c + ab)\,h = -\,b;$$

and, similarly, $$(c + ab)\,k = -\,a:$$

hence $$\frac{h}{b} = \frac{k}{a},$$

which is the equation to the required locus: thus we see that the centre of the conic section lies always in a straight line passing through the intersection of the two given lines, and bisecting the chord which joins the two given points of contact.

Gergonne: *Annales de Mathématiques*, tom. XI. p. 385.

4. If a conic section be touched by four straight lines, to find the locus of its centre.

If we take two of the tangents as axes of coordinates, the equation to the conic section will be of the form

$$a^2x^2 + 2cxy + b^2y^2 + 2ax + 2by + 1 = 0.$$

The equation to the tangent, at any point (x, y), is

$$(a^2x + cy + a)\,x' + (b^2y + cx + b)\,y' + ax + by + 1 = 0.$$

Let this tangent coincide with a line

$$\beta x' + \alpha y' = \alpha\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then $$\alpha\,(a^2x + cy + a) + ax + by + 1 = 0 \ldots\ldots\ldots\ldots(2),$$

$$\beta\,(b^2y + cx + b) + ax + by + 1 = 0 \ldots\ldots\ldots\ldots(3).$$

Eliminating x from (1), (2), (3), we shall arrive at an equation

$$\alpha\beta\,(ab + c) + 2\,(a\alpha + b\beta + 1) = 0 \ldots\ldots\ldots\ldots(4).$$

For another tangent, we shall have, similarly,

$$\alpha'\beta'\,(ab + c) + 2\,(a\alpha' + b\beta' + 1) = 0 \ldots\ldots\ldots(5).$$

But, if x, y, be the coordinates of the centre of the conic section,

$$a^2x + cy + a = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$b^2y + cx + b = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Eliminating a, b, c, from (4), (5), (6), (7), we shall get

$$\frac{x}{\alpha - \alpha'} + \frac{y}{\beta - \beta'} = \frac{\alpha\beta - \alpha'\beta'}{2\,(\alpha - \alpha')\,(\beta - \beta')} \ldots\ldots\ldots (8)$$

for the equation to the locus of the centre, which is therefore a straight line.

Cor. It may easily be proved that this line passes through the middle points of the two diagonals of the quadrilateral formed by the four tangents of the conic section.

Newton: *Principia*, lib. I. Lemma, XXV. Cor. 3.
Gergonne: *Annales de Mathématiques*, tom. XI. p. 382, *ann.* 1820, 1821.
Poncelet: *Gergonne, Annales de Mathématiques*, tom. XII. p. 109.
Eligius Manderlier: *Annales Academiæ Gandavensis*, 1826-28, p. 57.
Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 362.
Hearn: *Researches on Curves of the Second Order*, p. 37.

5. From a point O, two straight lines are drawn, touching a conic section in H and K. Two straight lines EPP', EQQ', are drawn, cutting each other in E, the former parallel to OH, the latter to OK; the former cutting the curve in P, P', the latter cutting it in Q, Q'. To prove that

$$EP.EP' : EQ.EQ' :: (OH)^2 : (OK)^2.$$

If OH, OK, produced indefinitely, be chosen as axes of x, y, respectively, the equation to the conic section will be

$$\frac{x^2}{a^2} + \frac{2xy}{mn} + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0,$$

where a, b, denote OH, OK, respectively, m, n, being any constants.

Let h, k, be the coordinates of E: then, taking EPP', EQQ', produced indefinitely, as axes of x', y', respectively, we shall have, for the equation to the curve,

$$\frac{(x'+h)^2}{a^2} + \frac{2(x'+h)(y'+k)}{mn} + \frac{(y'+k)^2}{b^2} - \frac{2(x'+h)}{a} - \frac{2(y'+k)}{b} + 1 = 0.$$

Putting $x' = 0$, the two values of y' in the resulting quadratic will be EQ, EQ', and accordingly

$$EQ.EQ' = b^2\left(\frac{h^2}{a^2} + \frac{2hk}{mn} + \frac{k^2}{b^2} - \frac{2h}{a} - \frac{2k}{b} + 1\right).$$

By symmetry we have also

$$EP.EP' = a^2\left(\frac{h^2}{a^2} + \frac{2hk}{mn} + \frac{k^2}{b^2} - \frac{2h}{a} - \frac{2k}{b} + 1\right).$$

Hence $$\frac{EP.EP'}{EQ.EQ'} = \frac{(OH)^2}{(OK)^2}.$$

De la Hire: *Sectiones Conicæ*, lib. III. prop. 29.

6. To find the condition that the straight line

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1$$

may touch a conic section represented by the equation

$$\frac{x^2}{a^2} + 2\frac{x}{m}.\frac{y}{n} + \frac{y^2}{b^2} + 2\frac{x}{a} + 2\frac{y}{b} + 1 = 0.$$

The required condition is expressed by the equation

$$\alpha\beta\left(\frac{1}{ab} + \frac{1}{mn}\right) + 2\left(\frac{\alpha}{a} + \frac{\beta}{b} + 1\right) = 0.$$

7. From a point O, two straight lines are drawn, touching a conic section in H and K. A straight line $MQRT$ is drawn, parallel to one of these tangents, cutting the other in M, the curve in Q, T, and the chord HK in R. To prove that

$$MQ.MT = (MR)^2.$$

De la Hire: *Sectiones Conicæ*, lib. III. prop. 27.

8. To find the locus of the centres of all conic sections, which touch three given straight lines, and pass through a given point.

If two of the given straight lines be taken as the axes of coordinates; then (h, k) being the given point, and $\frac{x}{\alpha} + \frac{y}{\beta} = 1$, the equation to the third straight line, the equation to the required locus will be

$$4hk(2x - h)(2y - k) = \{2(kx + hy - hk) - (2\beta x + 2\alpha y - \alpha\beta)\}^2.$$

Gergonne: *Annales de Mathématiques*, tom. XI. p. 385.
Poncelet: *Gergonne, Annales de Mathématiques*, tom. XII. p. 111.

9. To determine the locus of the centres of all the conic sections, which touch two given straight lines and pass through two given points.

The two straight lines being taken as coordinate axes, and (h, k), (h', k'), being the two given points; the equation to the required locus will be

$$\{(hy - kx)^2 - (h'y - k'x)^2\}^2$$
$$= 4\{(kx + hy - hk) - (k'x + h'y - h'k')\}.\{(hy - kx)^2.(k'x + h'y - h'k')$$
$$- (h'y - k'x)^2.(kx + hy - hk)\}.$$

This equation may be resolved into a double equation of the second degree. If we put

$$(hk' - kh')(hk - h'k') = M,$$

the double equation will be

$$(k+k').[(h-h')^2.\{(k+k')x - (h+h')y\}^2 - 2Mx\{2x - (h+h')\}].(kk')^{\frac{1}{2}}$$
$$= \pm(h+h').[(k-k')^2.\{(k+k')x - (h+h')y\}^2 + 2My\{2y - (k+k')\}].(hh')^{\frac{1}{2}}.$$

Brianchon: *Gergonne, Annales de Mathématiques*, tom. XI. p. 219.
Gergonne: *Annales de Mathématiques*, tom. XI. p. 390.
Poncelet: *Gergonne, Annales de Mathématiques*, tom. XII. p. 233.
Gergonne: *Annales de Mathématiques*, tom. XII. p. 249.

10. To prove that the straight line which joins the summit of an angle circumscribed about a conic section with the centre of the curve, divides the chord of contact into two equal parts.

Frédéric Sarrus: *Gergonne, Annales de Mathématiques*, tom. XII. p. 368.

SECTION VI.

Referred to a Tangent and Normal.

1. If, in any conic section, be inscribed a series of right-angled triangles, having all of them the summit of the right angle at a given point of the curve; to prove that their hypotenuses will all meet at a single point of the normal drawn through the common summit of all these triangles.

Let the tangent and normal at the given point be taken as axes of x and y respectively. Then the equation to the conic section will be

$$ax^2 + by^2 + 2cxy + 2fy = 0.$$

The equations to the two sides of one of the right angles being

$$y - \alpha x = 0, \quad y + \frac{1}{\alpha} x = 0,$$

the equation to both will be

$$(y - \alpha x)\left(y + \frac{1}{\alpha} x\right) = 0,$$

whence $$x^2 = y^2 + \left(\frac{1}{\alpha} - \alpha\right) xy.$$

Substituting this expression for x^2 in the equation to the conic section, we get, dividing by y, for the equation to the third side, $$\left\{a\left(\frac{1}{\alpha} - \alpha\right) + 2c\right\} x + (a + b)\, y + 2f = 0.$$

Putting $x = 0$, we see that

$$y = -\frac{2f}{a + b}.$$

Thus we see that the hypotenuse cuts the normal at an invariable distance from the origin.

Cor. It hence follows by the theory of poles and polars, that the points of concourse of the tangents at the extremities of these hypotenuses lie all in a single straight line.

Frégier: *Gergonne*, *Annales de Mathématiques*, tom. VI. p. 231.

2. If, in any conic section, be inscribed a series of triangles, all having a common summit, the angle of which is bisected by the normal to the curve at the place of the summit: to prove that the sides of these triangles, opposite to this summit, will all meet in the point of intersection of the tangents drawn at the two ends of the normal.

The equation to the conic section, referred to the tangent and normal, at the common summit, as axes of coordinates, will be $$ax^2 + by^2 + 2cxy + 2fy = 0 \;\ldots\ldots\ldots\ldots\ldots (1).$$

Also the equation to the two sides, of any one of the triangles, which pass through the origin, will be

$$x^2 = \lambda y^2 \ldots\ldots\ldots\ldots\ldots (2).$$

Substituting in the equation (1) the expression for x^2 given by (2), and dividing by y, we get, for the equation to the third side,

$$(a\lambda + b)\, y + 2\,(cx + f) = 0 \ldots\ldots\ldots\ldots (3).$$

Putting $y = 0$, we see that

$$x = -\frac{f}{c},$$

a value independent of λ. Hence the opposite sides of all the triangles intersect the tangent at a distance $-\frac{f}{c}$ from the point of contact. The tangent at the other end of the normal, as coinciding with the chord (3), when λ becomes zero, must also pass through the same point.

Cor. Hence it follows, by the theory of poles, that the points of concourse of the tangents at the extremities of these third sides of the triangles, will all lie in a single straight line, viz. the normal itself.

Frégier: *Gergonne*, *Annales de Mathématiques*, tom. VI. p. 233.

3. To find the equation to a conic section which has a contact of the third order with the curve

$$ax^2 + by^2 + 2cxy + 2fy = 0 \ldots\ldots\ldots\ldots (1)$$

at the origin of coordinates.

The equation to a conic section, which has a contact of the first order with (1) at the origin, is

$$a_{\prime}x^2 + b_{\prime}y^2 + 2c_{\prime}xy + 2f_{\prime}y = 0 \ldots\ldots\ldots\ldots (2).$$

At the intersections of (1) and (2), we have

$$(a_{\prime}b - ab_{\prime})\, y + 2\,(a_{\prime}c - ac_{\prime})\, x + 2\,(a_{\prime}f - af_{\prime}) = 0 \ldots (3).$$

If the straight line (3), which joins the two points of intersection, pass through the origin, the curves (1) and (2) will have two elements in common at the origin, or the contact will be of the second order: the corresponding condition being

$$a_{\prime}f = af_{\prime} \ldots\ldots\ldots\ldots\ldots (4).$$

Again, if the line (3) coalesce with the axis of x, the remaining point of intersection will coalesce with the origin, and thus, the two curves having three elements in common at the origin, the contact will be of the third order. The additional relation is

$$a_{,}c = ac_{,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The relations (4) and (5) reduce the equation (2) to the form

$$ax^2 + \frac{b_{,}f}{f_{,}}y^2 + 2cxy + 2fy = 0,$$

which is the required equation.

Plucker: *Gergonne*, *Annales de Mathématiques*, tom. XVII. p. 69.

4. If two conic sections touch each other at any point, to prove that they cannot have more than two points of intersection.

’Εὰν τῶν εἰρημένων γραμμῶν τινες καθ’ ἓν ἐφάπτωνται σημεῖον ἀλλήλων· οὐ συμβάλλουσιν ἑαυταῖς καθ’ ἕτερα σημεῖα πλείονα ἢ δύο.

ΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ Κωνικῶν τὸ τέταρτον· Πρότασις κς′.

5. To prove that, for every point of a conic section, there exists a circle which has with the curve, at this point, a contact of the second order.

Plucker: *Gergonne*, *Annales de Mathématiques*, tom. XVII. p. 71.

6. To prove that a circle cannot have with a line of the second order, a contact of the third order at one of its points, unless this point be at the end of an axis of the curve.

Plucker: *Gergonne*, *Annales de Mathématiques*, tom. XVII. p. 71.

7. To prove that, if any number of circles touch a conic section at the same point, the chords joining the points of intersection are all parallel to each other.

Plucker: *Gergonne*, *Annales de Mathématiques*, tom. XVII. p. 71.

SECTION VII.

Referred to any Axes whatever. Centres.

If α, β, be the coordinates of the centre of the curve

$$\phi(x, y) = ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

to find the value of $\phi(\alpha, \beta)$.

Changing the origin of coordinates to the point α, β, we shall have for our transformed equation,

$$a(x' + \alpha)^2 + b(y' + \beta)^2 + 2c(x' + \alpha)(y' + \beta) + 2a'(x' + \alpha) + 2b'(y' + \beta) + c' = 0.$$

Since (α, β) is the centre, the coefficients of x' and y' must be zero: hence

$$a\alpha + c\beta + a' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$b\beta + c\alpha + b' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$a\alpha^2 + b\beta^2 + 2c\alpha\beta + a'\alpha + b'\beta = 0,$$

and therefore

$$\phi(\alpha, \beta) = c' + a'\alpha + b'\beta \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2) we may easily get

$$\alpha = \frac{a'b - b'c}{c^2 - ab}, \qquad \beta = \frac{b'a - a'c}{c^2 - ab};$$

and therefore, from (3),

$$\phi(\alpha, \beta) = c' + \frac{ab'^2 + ba'^2 - 2a'b'c}{c^2 - ab}.$$

SECTION VIII.

Referred to any Axes whatever. Tangents.

1. To find the condition that the straight line

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

may touch the conic section

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0.$$

If x', y', be the coordinates of the point of contact, the equation to the tangent is

$$(ax' + cy' + a')\,x + (by' + cx' + b')\,y + a'x' + b'y' + c' = 0.$$

Since this equation must coincide with the equation (1), we must have

$$\alpha\,(ax' + cy' + a') + a'x' + b'y' + c' = 0,$$

and $$\beta\,(by' + cx' + b') + a'x' + b'y' + c' = 0;$$

or $$(\alpha a + a')\,x' + (\alpha c + b')\,y' + \alpha a' + c' = 0,$$

and $$(\beta b + b')\,y' + (\beta c + a')\,x' + \beta b' + c' = 0.$$

But, since the point of contact lies in the line (1), we must have also $$\beta x' + \alpha y' - \alpha\beta = 0.$$

Eliminating x' and y' between the last three equations, we shall obtain for the required condition,

$$\alpha^2\beta^2\,(c^2 - ab) + 2\alpha^2\beta\,(a'c - ab') + 2\alpha\beta^2\,(b'c - ba')$$
$$+ \alpha^2\,(a'^2 - ac') + \beta^2\,(b'^2 - bc') + 2\alpha\beta\,(cc' - a'b') = 0.$$

Gergonne: *Annales de Mathématiques*, tom. XI. p. 381.

2. To prove that a conic section represented by the equation

$$(lu)^{\frac{1}{2}} + (mv)^{\frac{1}{2}} + (nw)^{\frac{1}{2}} = 0 \,\ldots\ldots\ldots\ldots\ldots\, (1),$$

where u, v, w, are linear functions of x and y, and l, m, n, are arbitrary constants, is touched by a straight line

$$\lambda u + \mu v + \nu w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

provided that l, m, n, λ, μ, ν, are connected together by the equation

$$\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating ν between (2) and (3), we get

$$nw = \left(\frac{l}{\lambda} + \frac{m}{\mu}\right).(\lambda u + \mu v) \ldots\ldots\ldots\ldots\ldots (4).$$

Combining (1) and (4), we have

$$(lu)^{\frac{1}{2}} + (mv)^{\frac{1}{2}} + \left(\frac{l}{\lambda} + \frac{m}{\mu}\right)^{\frac{1}{2}}.(\lambda u + \mu v)^{\frac{1}{2}} = 0,$$

and therefore

$$lu + mv + 2\,(lmuv)^{\frac{1}{2}} = \left(\frac{l}{\lambda} + \frac{m}{\mu}\right)(\lambda u + \mu v),$$

$$2\,(lmuv)^{\frac{1}{2}} = \frac{l\mu}{\lambda}\,v + \frac{m\lambda}{\mu}\,u,$$

$$\left(\frac{l\mu v}{\lambda}\right)^{\frac{1}{2}} - \left(\frac{m\lambda u}{\mu}\right)^{\frac{1}{2}} = 0,$$

whence

$$\frac{\lambda^2 u}{l} = \frac{\mu^2 v}{m};$$

and therefore, by similarity,

$$\frac{\lambda^2 u}{l} = \frac{\mu^2 v}{m} = \frac{\nu^2 w}{n}.$$

These two equations shew that the lines (1) and (2) meet in one point and one only: hence (2) is a tangent to (1).

It may be observed that, since the equation (2) involves two arbitrary parameters, viz. $\frac{\lambda}{\mu}$, $\frac{\lambda}{\nu}$, subject to only one restriction, viz. (3), the tangent line is a general tangent to the conic section.

Cor. If $\mu = 0$ and $\nu = 0$, then the equation (2) is reduced to $u = 0$.

Thus it appears that $u = 0$, and, similarly, $v = 0$, $w = 0$, are all tangents to (1).

The truth of this conclusion is easily seen also by combining $u = 0$ with (1), whereby we perceive at once that the two lines meet in one and only one point.

Hearn: *Researches on Curves of the Second Order*, p. 36.

3. To find the equation to the tangent at any point of a conic section represented by the equation

$$uv = w^2 \text{..............................}(1),$$

where u, v, w, are linear functions of x and y.

The equations to any point in the curve may evidently be written in the form

$$\lambda u = w, \qquad v = \lambda w,$$

λ being an arbitrary quantity.

The equation to any straight line passing through this point may be expressed in the form

$$v - \lambda w = \mu (\lambda u - w) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

μ being an arbitrary quantity.

At the intersection of (1) and (2), we have, eliminating v,

$$w^2 - \lambda uw = \mu (\lambda u^2 - uw),$$

or

$$w^2 - (\lambda - \mu) uw - \lambda\mu u^2 = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

Suppose this equation to be a perfect square: then

$$(\lambda - \mu)^2 = - 4\lambda\mu,$$

$$(\lambda + \mu)^2 = 0,$$

$$\mu = - \lambda;$$

and therefore (2) becomes

$$\lambda^2 u - 2\lambda w + v = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since (3) is a perfect square, (4) meets (1) in only one point, that is, at the point $v = \lambda w$, $\lambda u = w$: hence (4) is the equation to the required tangent.

Salmon: *Conic Sections*, p. 217.

4. To prove that $\lambda u + \mu v + \nu w = 0$

will be always a tangent to a conic section

$$lvw + mwu + nuv = 0,$$

where u, v, w, are linear functions of u, v, w, and l, m, n, are parameters, provided that

$$(l\lambda)^{\frac{1}{2}} + (m\mu)^{\frac{1}{2}} + (n\nu)^{\frac{1}{2}} = 0.$$

Hearn: *Researches on Curves of the Second Order*, p. 34.

SECTION IX.

Referred to any Axes whatever. Chords.

1. To determine the locus of the middle points of the chords of the curve

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0,$$

which are parallel to the line

$$x \sin\theta - y \cos\theta = 0;$$

and thence, the axes being rectangular, to find the positions of the axes of the curve.

Let h, k, be the coordinates of the middle point of any one of the parallel chords: let x', y', be the coordinates of either end of this chord referred to (h, k) as origin. Then

$$a(h+x')^2+b(k+y')^2+2c(h+x')(k+y')+2a'(h+x')+2b'(k+y')+c'=0.$$

Now for x', y', we may write equal negative values $-x'$, $-y'$, because the chord is bisected in the new origin: hence the sum of the terms involving the first powers of x', y', must be equal to zero, and thus

$$(ah + ck + a')\,x' + (bk + ch + b')\,y' = 0.$$

But this chord is parallel to the line

$$x \sin\theta - y \cos\theta = 0:$$

hence $$(ah + ck + a') \cos\theta + (bk + ch + b') \sin\theta = 0,$$

which is the equation to the required locus.

Cor. Since an axis of the curve intersects its chords at right angles, we shall have, for the determination of the positions of the axes, supposing the coordinate axes to be rectangular,

$$\tan\theta = \frac{b \sin\theta + c \cos\theta}{a \cos\theta + c \sin\theta},$$

whence $$a \sin 2\theta + c\,(1 - \cos 2\theta) = b \sin 2\theta + c\,(1 + \cos 2\theta),$$

$$(a - b) \tan 2\theta = 2c,$$

$$\tan 2\theta = \frac{2c}{a - b}.$$

2. In any conic section, if two chords PQ, PR, make equal angles with a fixed chord PK, and the chord QR be drawn: to prove that QR will pass through a fixed point for all positions of PQ, PR.

Let PK, produced indefinitely, be taken as the axis of y, and Px, at right angles to it, as the axis of x. The equation to the conic section will be

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y = 0 \text{..........}(1).$$

Let the equation to PQ be $y = mx$; that of PR will be $y = -mx$, and therefore both PQ and PR will be represented by the equation

$$y^2 = m^2x^2 \text{..........................}(2).$$

From (1) we have

$$ax^2 + by^2 + 2cxy + 2a'x = -2b'y \text{...........}(3),$$

and
$$ax^2 + by^2 + 2cxy = -2a'x - 2b'y \text{...........}(4),$$

and therefore, multiplying the equation (3) by $a'x$, the equation (4) by $b'y$, and subtracting the latter of the resulting equations from the former, we get

$$a'x\,(ax^2 + by^2 + 2cxy + 2a'x) - b'y\,(ax^2 + by^2 + 2cxy) = 2b'^2y^2 \text{...}(5).$$

At the intersections of QR with the curve, we have, by (2) and (5),

$$a'\,(ax + m^2bx + 2cy + 2a') - b'\,(ay + m^2by + 2m^2cx) = 2m^2b'^2,$$

which is therefore the equation to the chord QR.

This equation is evidently satisfied, for all values of m, by the two equations

$$aa'x + (2a'c - ab')\,y + 2a'^2 = 0,$$

$$(a'b - 2b'c)\,x - bb'y - 2b'^2 = 0,$$

which are the equations to a fixed point.

3. To find the equation to a chord passing through any two proposed points of the conic section represented by the equation

$$uv = w^2,$$

u, v, w, being linear functions of x and y.

Let the two points be

$$\left\{\begin{aligned}\lambda u &= w\\ v &= \lambda w\end{aligned}\right\} \text{ and } \left\{\begin{aligned}\lambda' u &= w\\ v &= \lambda' w\end{aligned}\right\},$$

λ, λ', being any proposed constants.

The equation to any straight line through the former point is

$$v - \lambda w = \rho\,(\lambda u - w),$$

and that to any straight line through the latter point is

$$v - \lambda' w = \rho'\,(\lambda' u - w),$$

ρ and ρ' being arbitrary parameters.

Now these two straight lines will coincide provided that

$$\rho\lambda = \rho'\lambda', \qquad \rho - \lambda = \rho' - \lambda' :$$

whence $$\rho = -\lambda', \qquad \rho' = -\lambda.$$

The equation to either of the lines then becomes

$$\lambda\lambda' u - (\lambda + \lambda')\,w + v = 0.$$

Since this line passes through both the proposed points, it is the equation to the chord.

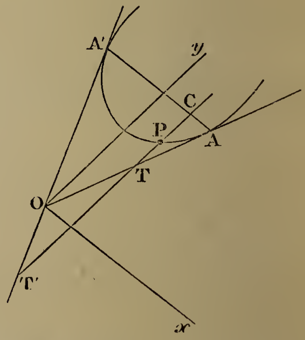
Cor. If $\lambda' = \lambda$, the two points coincide and the chord becomes a tangent represented by the equation

$$\lambda^2 u - 2\lambda w + v = 0.$$

Salmon: *Conic Sections*, p. 217.

4. Through any point P in a conic section is drawn a straight line, parallel to a given straight line, intersecting two tangents in T and T', and cutting the chord, which joins the points of contact, in C. To prove that

$$(PC)^2 \propto PT.PT'.$$



Let O be the intersection of the two tangents. Let the axes of coordinates be Ox, parallel to the chord of contact $A'A$, and Oy, parallel to the line $T'TPC$.

Then, the equations to OA, OA', AA', being represented by

$$y = \alpha x, \quad y = -\alpha' x, \quad y = b,$$

that to the conic section will be, λ denoting an arbitrary constant quantity,

$$(y - \alpha x)(y + \alpha' x) = \lambda\,(y - b)^2,$$

which, expressed geometrically, becomes

$$PT.PT' = \lambda (PC)^2,$$

whence $$(PC)^2 \propto PT.PT'.$$

Carnot: *Géométrie de Position*, p. 446.

5. If APP' be any chord, drawn through a fixed point A in the axis of a conic section, to meet it in two points P, P'; to prove that, A being the origin of coordinates and the axis of the conic section the axis of x,

$$\frac{1}{x_{\prime}} + \frac{1}{x_{\prime\prime}} = \text{a constant quantity},$$

$x_{\prime}$, $x_{\prime\prime}$, being the abscissæ of P, P'.

6. If two conic sections are such that they intercept, on a given straight line, chords the middle points of which are coincident: to prove that the same property will apply to all conic sections which pass through the four intersections of the two conic sections and cut the same given line.

Plucker: *Gergonne, Annales de Mathématiques*, tom. XIX. p. 105.

SECTION X.

Referred to any Axes whatever. Directrix.

1. To find the equation to a conic section, of which (h, k) is the focus, e the eccentricity, and of which the directrix, the axes of coordinates being rectangular, is represented by the equation

$$x \sin\alpha - y \cos\alpha + c = 0;$$

and, if h, k, be such that

$$h + e^2 c \sin\alpha = 0, \quad k - e^2 c \cos\alpha = 0,$$

to find the equation to the curve referred to its axes.

The equation to the curve, referred to the existing axes, is

$$(x-h)^2 + (y-k)^2 = e^2 (x \sin\alpha - y \cos\alpha + c)^2;$$

and, the two relations between h, k, α, e, c, being adopted, the equation to the curve, referred to its axes, is

$$x^2 + (1-e^2) y^2 = e^2 c^2 (1-e^2).$$

2. To find the equation to the directors of the conic section

$$ax^2 + by^2 + 2cxy = f.$$

The required equation to the two directors is

$$\{(a+\lambda)^{\frac{1}{2}}x + (b+\lambda)^{\frac{1}{2}}y\}^2 = \frac{f\lambda}{a+b+\lambda},$$

λ being given by the equation

$$(a+\lambda)(b+\lambda) = c^2.$$

SECTION XI.

Referred to any Axes whatever. Conjoint Lines and Circles.

1. Having given the equation to a conic section, and that to a straight line in the same plane, to find the equation to the *conjoint line* which passes through the origin, and the equation to the corresponding *conjoint circle.*

The expression *conjoint lines* is used by M. Terquem to signify two straight lines such that, if they be taken as axes of coordinates, the coefficients of the two squares in the equation to the curve become equal. Thus two equal diameters are *conjoint lines;* the same may be said of the asymptotes of an hyperbola: a principal diameter coalesces with its *conjoint line.*

It is easily seen that two conjoint lines cut a conic section, generally speaking, in four points, situated in a single circle, called a *conjoint circle;* and, reciprocally, that if four points of a conic section lie in a circle, any two lines joining them two and two, are conjoint lines.

Let the equation to any conic section be

$$ax^2 + 2cxy + by^2 + 2a'x + 2b'y + c' = 0 \;\ldots\ldots\ldots (1),$$

and let γ be the angle between the coordinate axes.

Let the equation to the given straight line be

$$\lambda x + \mu y + \nu = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let the equation to the required *conjoint line* be

$$\lambda' x + \mu' y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The equation to the system of the two straight lines will be

$$(\lambda x + \mu y + \nu)(\lambda' x + \mu' y) = 0 \ldots\ldots\ldots\ldots\ldots (4).$$

Adding together the equations (1) and (4), we have

$$(a + \lambda\lambda') x^2 + (2c + \lambda\mu' + \lambda'\mu) xy + (b + \mu\mu') y^2$$
$$+ (2a' + \lambda'\nu) x + (2b' + \mu'\nu) y + c' = 0 \ldots\ldots (5).$$

In order that this equation may represent a circle, we must have

$$a + \lambda\lambda' = b + \mu\mu',$$
$$2c + \lambda\mu' + \lambda'\mu = 2\cos\gamma\,(a + \lambda\lambda'),$$

whence we deduce

$$\left.\begin{aligned} \lambda' &= \frac{\lambda(b-a) + 2\mu(a\cos\gamma - c)}{\delta^2} \\ \text{and}\qquad \mu' &= \frac{\mu(a-b) + 2\lambda(b\cos\gamma - c)}{\delta^2} \\ \text{where}\qquad \delta^2 &= \lambda^2 - 2\lambda\mu\cos\gamma + \mu^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots (6).$$

Substituting the values (6) in the equation (5), and putting

$$\mu^2 a - 2\lambda\mu c + \lambda^2 b = k,$$

we shall obtain

$$kx^2 + 2k\cos\gamma . xy + ky^2 + \{2a'\delta^2 + \lambda\nu(b-a) + 2\mu\nu(a\cos\gamma - c)\}\,x$$
$$+ \{2b'\delta^2 + \mu\nu(a-b) + 2\lambda\nu(b\cos\gamma - c)\}\,y + c'\delta^2 = 0,$$

which is the equation to the conjoint circle.

Terquem: *Liouville, Journal de Mathématiques*, tom. III. p. 17.

2. To find the locus of the centre of the conjoint circle of a given conic section, the two conjoint lines being both supposed always to pass through the origin.

If the equation to the conic section be

$$ax^2 + 2cxy + by^2 + 2a'x + 2b'y + c' = 0,$$

the required locus will be a straight line of which the equation is

$$(b' - a'\cos\gamma)\,x = (a' - b'\cos\gamma)\,y.$$

Terquem: *Liouville, Journal de Mathématiques*, tom. III. p. 18.

SECTION XII.

Passing through given Points.

1. To determine the equation to a conic section which passes through the five points of which the coordinates are

$$1,\ -1;\quad 2,\ 1;\quad -2,\ 3;\quad 3,\ 2;\quad -1,\ -3.$$

If $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, be the equations to the sides of the quadrilateral formed by joining in order the first four points, the equation to the conic section will be

$$\alpha\gamma = \lambda\beta\delta,$$

where λ = a constant. By giving to x, y, their values at the fifth point, we shall, by this equation, determine λ.

Now, by the formula

$$(x_2 - x_1)\,y - (y_2 - y_1)\,x - x_2y_1 + x_1y_2 = 0,$$

we see that

$$\alpha=y-2x+3,\quad \beta=-4y-2x+8,\quad \gamma=5y+x-13,\quad \delta=-2y+3x-5.$$

Hence the equation to the conic section will be

$$(y - 2x + 3)(5y + x - 13) = 2\lambda(2y + x - 4)(2y - 3x + 5).$$

Putting $x=-1$, $y=-3$, we have

$$-58 = -2\lambda \times 22,\qquad 2\lambda = \tfrac{29}{11}.$$

Hence the equation to the curve becomes

$$11(y - 2x + 3)(5y + x - 13) = 29(2y + x - 4)(2y - 3x + 5),$$

or $$61y^2 - 17xy - 65x^2 + 36y + 174x - 151 = 0.$$

2. To find the locus of the centre of a conic section which always passes through four given points.

Let A, A', B, B', be the four points: join AA', BB', O being their point of intersection. Take OA, OB, produced indefinitely, as axes of x, y.

Let $OA=\alpha$, $OA'=\alpha'$, $OB=\beta$, $OB'=\beta'$.

Then the equation to the conic section will be

$$\frac{x^2}{\alpha\alpha'} - cxy + \frac{y^2}{\beta\beta'} + \frac{x}{\alpha\alpha'}(\alpha' - \alpha) + \frac{y}{\beta\beta'}(\beta' - \beta) = 1.$$

Let X, Y, be the coordinates of its centre: then

$$\frac{2X}{\alpha\alpha'} - cY + \frac{1}{\alpha\alpha'}(\alpha' - \alpha) = 0,$$

$$\frac{2Y}{\beta\beta'} - cX + \frac{1}{\beta\beta'}(\beta' - \beta) = 0.$$

Eliminating c, we have, for the locus of the centres,

$$\frac{2X^2}{\alpha\alpha'} - \frac{2Y^2}{\beta\beta'} + \frac{X}{\alpha\alpha'}(\alpha' - \alpha) - \frac{Y}{\beta\beta'}(\beta' - \beta) = 0 \ldots\ldots(1),$$

which is the equation to an ellipse or hyperbola accordingly as $\alpha\alpha'$ and $\beta\beta'$ have different or like signs.

Cor. 1. The equations to the asymptotes of the hyperbola may be shewn to be

$$\frac{y}{(\beta\beta')^{\frac{1}{2}}} \mp \frac{x}{(\alpha\alpha')^{\frac{1}{2}}} = \pm \tfrac{1}{4}\left\{\frac{\alpha' - \alpha}{(\alpha\alpha')^{\frac{1}{2}}} \mp \frac{\beta' - \beta}{(\beta\beta')^{\frac{1}{2}}}\right\}.$$

Cor. 2. It is easily seen that the centre of the curve (1) coincides with the middle point of the line joining the bisections of AA', BB'.

Brianchon: *Gergonne, Annales de Mathématiques*, tom. XI. p. 219.
Gergonne: *Annales de Mathématiques*, tom. XI. p. 396.
Poncelet: *Gergonne, Annales de Mathématiques*, tom. XII. p. 245.

The following is a different solution of the same problem.

Let the equations to the sides of the triangle formed by joining, two and two, three of the four given points be

$$u_1 = x\cos\alpha_1 + y\sin\alpha_1 - \delta_1 = 0,$$

$$u_2 = x\cos\alpha_2 + y\sin\alpha_2 - \delta_2 = 0,$$

$$u_3 = x\cos\alpha_3 + y\sin\alpha_3 - \delta_3 = 0.$$

Then the equation to the conic section will be of the form

$$\lambda_1 u_2 u_3 + \lambda_2 u_3 u_1 + \lambda_3 u_1 u_2 = 0 \ldots\ldots\ldots\ldots\ldots(1),^*$$

where λ_1, λ_2, λ_3, are arbitrary constants.

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* Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 320.

If we differentiate the equation (1) partially, first with regard to x and then with regard to y, we shall get

$$\lambda_1(u_2\cos\alpha_3+u_3\cos\alpha_2)+\lambda_2(u_3\cos\alpha_1+u_1\cos\alpha_3)+\lambda_3(u_1\cos\alpha_2+u_2\cos\alpha_1)=0,$$

$$\lambda_1(u_2\sin\alpha_3+u_3\sin\alpha_2)+\lambda_2(u_3\sin\alpha_1+u_1\sin\alpha_3)+\lambda_3(u_1\sin\alpha_2+u_2\sin\alpha_1)=0,$$

two equations which determine the values of x and y at the centre of the conic section.

Eliminating λ_3 between these two equations and effecting obvious simplifications, we shall get

$$\frac{\lambda_1}{\lambda_2}=\frac{u_1}{u_2}\cdot\frac{u_2\sin(\alpha_3-\alpha_1)+u_3\sin(\alpha_1-\alpha_2)-u_1\sin(\alpha_2-\alpha_3)}{u_3\sin(\alpha_1-\alpha_2)+u_1\sin(\alpha_2-\alpha_3)-u_2\sin(\alpha_3-\alpha_1)},$$

or, putting

$$u_1\sin(\alpha_2-\alpha_3)+u_2\sin(\alpha_3-\alpha_1)+u_3\sin(\alpha_1-\alpha_2)=2v,$$

$$\frac{\lambda_1}{\lambda_2}=\frac{u_1}{u_2}\cdot\frac{v-u_1\sin(\alpha_2-\alpha_3)}{v-u_2\sin(\alpha_3-\alpha_1)}.$$

By similarity we have also

$$\frac{\lambda_1}{\lambda_3}=\frac{u_1}{u_3}\cdot\frac{v-u_1\sin(\alpha_2-\alpha_3)}{v-u_3\sin(\alpha_1-\alpha_2)}.$$

Let m_1, m_2, m_3, be the values of u_1, u_2, u_3, at the fourth given point. Then, from (1),

$$\frac{\lambda_1}{m_1}+\frac{\lambda_2}{m_2}+\frac{\lambda_3}{m_3}=0,$$

and consequently, putting for λ_1, λ_2, λ_3, their proportionals, and writing ε_1, ε_2, ε_3, instead of $\sin(\alpha_2-\alpha_3)$, $\sin(\alpha_3-\alpha_1)$, $\sin(\alpha_1-\alpha_2)$, respectively, we have

$$v.\left(\frac{u_1}{m_1}+\frac{u_2}{m_2}+\frac{u_3}{m_3}\right)=\frac{\varepsilon_1.u_1^2}{m_1}+\frac{\varepsilon_2.u_2^2}{m_2}+\frac{\varepsilon_3.u_3^2}{m_3},$$

as the equation to the required locus, which is therefore a conic section.

This very elegant solution of the problem is given by Mr. Hearn, in his *Researches on Curves of the Second Order*, a work very remarkable for the ingenuity and beauty of its investigations.

3. If three conic sections pass through the same four points, their diameters, of which the conjugates are parallel to a single fixed line, all pass through one point.

Sturm: *Gergonne, Annales de Mathématiques*, tom. XVII. p. 176.
Lamé: *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*, p. 34.

SECTION XIII.

Passing through given Points and touching given straight lines.

1. To determine the locus of the centres of all conic sections which touch a given straight line and pass through three given points.

Let one of the three points be taken as the origin of coordinates, and let the axes pass through the other two points. Then, h, k, denoting the distances of these two points from the origin, the equation to the conic section will be

$$ax^2 + 2cxy + by^2 - ahx - bky = 0 \text{............}(1).$$

Let the equation to the given tangent line be

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1.$$

Now, generally, in order that a line expressed by this equation may touch a conic section denoted by the equation

$$ax^2 + 2cxy + by^2 + 2a'x + 2b'y + c' = 0,$$

we must have the condition

$$\alpha^2\beta^2(c^2 - ab) + 2\alpha^2\beta(a'c - ab') + 2\alpha\beta^2(b'c - ba')$$
$$+ \alpha^2(a'^2 - ac') + \beta^2(b'^2 - bc') + 2\alpha\beta(cc' - a'b') = 0.$$

In the present problem, where

$$a' = -\tfrac{1}{2}ah, \quad b' = -\tfrac{1}{2}bk, \quad c' = 0;$$

this condition reduces itself to

$$\tfrac{1}{4}(a\alpha h - b\beta k)^2 + \alpha\beta\{\alpha\beta(c^2 - ab) + a\alpha(bk - hc) + b\beta(ah - kc)\} = 0 \text{...}(2).$$

Now if x, y, be taken to represent the coordinates of the centre of (1), we shall have

$$2ax + 2cy - ah = 0,$$

and
$$2by + 2cx - bk = 0.$$

From these two equations we have

$$a = \frac{2cy}{h - 2x}, \qquad b = \frac{2cx}{k - 2y}.$$

Substituting these expressions for a and b in (2), we get

$$\begin{aligned}&\{\beta kx(2x-h) - \alpha hy(2y-k)\}^2\\ &+ \alpha\beta(2x-h)(2y-k).\{\alpha\beta(-2kx-2hy+hk)\\ &+ 2\alpha y(2kx+2hy-hk)\\ &+ 2\beta x(2hy+2kx-hk)\} = 0,\end{aligned}$$

or
$$\{\beta kx(2x-h) - \alpha hy(2y-k)\}^2$$
$$+ \alpha\beta(2x-h)(2y-k).(2kx+2hy-hk).(2\beta x+2\alpha y-\alpha\beta) = 0,$$

which is the equation to the required locus.

Gergonne: *Annales de Mathématiques*, tom. XI. p. 393.

The following is a different solution of the same problem. The notation of the second solution of Prob. (2) of the preceding section is retained.

Let the equation to the given line be put into the form

$$l_1u_1 + l_2u_2 + l_3u_3 = 0 \text{.....................(1)},$$

which may always be done by properly determining the constants l_1, l_2, l_3.

At the intersection of (1) and the equation to the conic section, viz.

$$\lambda_1u_2u_3 + \lambda_2u_3u_1 + \lambda_3u_1u_2 = 0,$$

we have $l_1\lambda_2u_1^2 + l_2\lambda_1u_2^2 + (l_1\lambda_1 + l_2\lambda_2 - l_3\lambda_3)\,u_1u_2 = 0;$

it is easily seen that this equation will be a perfect square, provided that λ_1, λ_2, λ_3, satisfy the condition

$$(l_1\lambda_1)^{\frac{1}{2}} + (l_2\lambda_2)^{\frac{1}{2}} + (l_3\lambda_3)^{\frac{1}{2}} = 0,$$

which is therefore the condition of contact.

Hence, replacing in this equation λ_1, λ_2, λ_3, by their proportionals, we have for the equation to the required locus,

$$\{l_1u_1(v-\varepsilon_1u_1)\}^{\frac{1}{2}}+\{l_2u_2(v-\varepsilon_2u_2)\}^{\frac{1}{2}}+\{l_3u_3(v-\varepsilon_3u_3)\}^{\frac{1}{2}}=0.$$

Hearn: *Researches on Curves of the Second Order.*

2. To find the locus of the centre of a conic section which touches four given straight lines.

Let the equations to three of the given lines be

$$\left.\begin{aligned} u_1 &= x\cos\alpha_1+y\sin\alpha_1-\delta_1=0\\ u_2 &= x\cos\alpha_2+y\sin\alpha_2-\delta_2=0\\ u_3 &= x\cos\alpha_3+y\sin\alpha_3-\delta_3=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$

The equation to the fourth line, c_1, c_2, c_3, being constants properly chosen, may be written in the form

$$c_1u_1+c_2u_2+c_3u_3=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

The equation to the conic section, which touches the lines (1), will be

$$(\lambda_1u_1)^{\frac{1}{2}}+(\lambda_2u_2)^{\frac{1}{2}}+(\lambda_3u_3)^{\frac{1}{2}}=0\ldots\ldots\ldots\ldots\ldots(3).$$

This conic section will also touch (2), provided that the parameters λ_1, λ_2, λ_3, are subject to the condition

$$\frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_2}+\frac{\lambda_3}{c_3}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4).$$

If we rationalize the equation (3), and then differentiate it, partially, first with regard to x, and secondly with regard to y, we shall obtain two equations for the determination of the values of x, y, at the centre of the conic section, viz. the equation

$$\begin{aligned} &\lambda_1^2u_1\cos\alpha_1+\lambda_2^2u_2\cos\alpha_2+\lambda_3^2u_3\cos\alpha_3\\ &=\lambda_2\lambda_3(u_2\cos\alpha_3+u_3\cos\alpha_2)+\lambda_3\lambda_1(u_3\cos\alpha_1+u_1\cos\alpha_3)\\ &\qquad\qquad+\lambda_1\lambda_2(u_1\cos\alpha_2+u_2\cos\alpha_1), \end{aligned}$$

and an equation which may be derived from this by writing *sin* throughout instead of *cos*.

From these two equations, by simple trigonometrical operations, we shall arrive at the following formulæ:

$$\frac{\varepsilon_3\lambda_2+\varepsilon_2\lambda_3}{u_1}=\frac{\varepsilon_1\lambda_3+\varepsilon_3\lambda_1}{u_2}=\frac{\varepsilon_2\lambda_1+\varepsilon_1\lambda_2}{u_3}\ldots\ldots\ldots(5),$$

where ε_1, ε_2, ε_3, are written for $\sin(\alpha_2-\alpha_3)$, $\sin(\alpha_3-\alpha_1)$, $\sin(\alpha_1-\alpha_2)$, respectively.

Putting each member of the double equality (5) equal to ρ, we shall get

$$\lambda_1 = \frac{\rho}{\varepsilon_2\varepsilon_3}(v-\varepsilon_1 u_1), \quad \lambda_2 = \frac{\rho}{\varepsilon_3\varepsilon_1}(v-\varepsilon_2 u_2), \quad \lambda_3 = \frac{\rho}{\varepsilon_1\varepsilon_2}(v-\varepsilon_3 u_3),$$

where
$$v = \tfrac{1}{2}(\varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3).$$

Substituting these values of λ_1, λ_2, λ_3, in the equation (4), we have, for the equation to the required locus,

$$\left(\frac{\varepsilon_1}{c_1}+\frac{\varepsilon_2}{c_2}+\frac{\varepsilon_3}{c_3}\right)v = \frac{\varepsilon_1^2 u_1}{c_1}+\frac{\varepsilon_2^2 u_2}{c_2}+\frac{\varepsilon_3^2 u_3}{c_3}.$$

Thus it appears that the locus is a straight line.

Hearn: *Researches on Curves of the Second Order.*

3. To find the locus of the centre of a conic section which passes through two given points ($u_1=0$, $u_3=0$) and ($u_2=0$, $u_3=0$), and touches a given straight line

$$u_1 + \gamma u_2 = 0$$

in a given point ($u_1 = 0$, $u_2 = 0$).

The locus required is a straight line defined by the equation

$$u_1\{v - u_1\sin(\alpha_2-\alpha_3)\} = \gamma u_2\{v - u_2\sin(\alpha_3-\alpha_1)\}.$$

Hearn: *Researches on Conic Sections*, p. 34.

4. To find the locus of the centre of a conic section which touches three given straight lines and passes through a given point.

The same notation being adopted as in problem (2), and m_1, m_2, m_3, represented the values of u_1, u_2, u_3, at the given point, the required locus will be a conic section defined by the equation

$$\{m_1\varepsilon_1.(v-\varepsilon_1 u_1)\}^{\frac{1}{2}} + \{m_2\varepsilon_2.(v-\varepsilon_2 u_2)\}^{\frac{1}{2}} + \{m_3\varepsilon_3.(v-\varepsilon_3 u_3)\}^{\frac{1}{2}} = 0.$$

Hearn: *Researches on Curves of the Second Order.*

5. To find the locus of the centre of a conic section which touches two given straight lines and passes through two given points.

Let (α, β), (α', β'), be the two given points: let the equations to the two given lines be

$$u = a_1x + b_1y + 1 = 0,$$
$$v = a_2x + b_2y + 1 = 0,$$

and that to the line joining the two given points

$$w = a_3x + b_3y + 1 = 0.$$

Also let w' represent $a_3'x + b_3'y$, a_3' and b_3' being quantities determined by the equations

$$w'_{\alpha,\,\beta} = a_3'\alpha + b_3'\beta = (uv)^{\frac{1}{2}}{}_{\alpha,\,\beta},$$
$$w'_{\alpha',\,\beta'} = a_3'\alpha' + b_3'\beta' = (uv)^{\frac{1}{2}}{}_{\alpha',\,\beta'}.$$

Then, putting

$$L = a_2b_3 - a_3b_2, \quad M = a_3b_1 - a_1b_3,$$
$$L' = a_2b_3' - a_3'b_2, \quad M' = a_3'b_1 - a_1b_3',$$

and
$$Q = a_3'b_3 - a_3b_3',$$

we shall have, for the equation to the required locus,

$$(Lu - Mv)^2 + 2Q\{(L'u - M'v)w - (Lu - Mv)w'\} = 0,$$

an equation of the second order.

Hearn: *Researches on Curves of the Second Order*, p. 42.

SECTION XIV.

Determination of their Equations from given Conditions.

1. To find the equation to the conic section which passes through the point (h, k) and touches the parabola

$$y^2 = lx$$

at the vertex and at an extremity of the latus-rectum.

The equation to the chord joining an extremity of the latus-rectum to the vertex of the parabola is

$$y - 2x = 0.$$

Hence the equation to the conic section, passing through both intersections of the lines

$$y^2 - lx = 0, \quad y - 2x = 0,$$

must be of the form

$$y^2 - lx + (y - 2x)(\lambda x + \mu y + \nu) = 0,$$

λ, μ, ν, being arbitrary constants.

But, since the conic section touches the parabola at the vertex, $\frac{x}{y} = 0$, when $x = 0$ and $y = 0$: hence $\nu = 0$. The equation therefore becomes

$$y^2 - lx + (y - 2x)(\lambda x + \mu y) = 0.$$

Also, since the conic section touches the parabola at the end of the latus-rectum, we must have $\frac{dx}{dy} = 1$, when $x = \frac{1}{4}l$, and $y = \frac{1}{2}l$: hence

$$2\mu + \lambda = 0:$$

thus the equation becomes

$$y^2 - lx = -\mu(y - 2x)^2.$$

But the conic section passes through the point h, k: hence

$$k^2 - lh = -\mu(k - 2h)^2.$$

Hence the equation to the conic section is

$$\frac{y^2 - lx}{k^2 - lh} = \left(\frac{y - 2x}{k - 2h}\right)^2.$$

Cor. Clearing the equation of fractions we shall get

$$y^2(4h^2 - 4hk + lh) + 4(k^2 - lh)xy - 4(k^2 - lh)x^2 \ldots = 0.$$

Hence the curve is an ellipse or an hyperbola as

$$(k^2 - lh)^2 < \text{or} > (4hk - 4h^2 - lh)(k^2 - lh),$$

or

$$(k^2 - lh)(k - 2h)^2 < \text{or} > 0.$$

Thus the curve will be an ellipse, if the point (h, k) is within, and an hyperbola, if it is without the parabola.

2. To find the equation to a conic section, having given that h, k, are the coordinates of its focus, that

$$x \cos\alpha + y \sin\alpha = \delta,$$

is the equation to its directrix, and that e is its eccentricity. Also, if h and k be such that

$$h = e^2\delta\cos\alpha, \quad k = e^2\delta\sin\alpha,$$

to find the axes of the conic section.

The equation to the conic section is

$$(x-h)^2 + (y-k)^2 = e^2(x\cos\alpha + y\sin\alpha - \delta)^2.$$

If $h = e^2\delta\cos\alpha$ and $k = e^2\delta\sin\alpha$, the semi-axes will be

$$e\delta, \quad (1-e^2)^{\frac{1}{2}}.e\delta,$$

or

$$e\delta, \quad (e^2-1)^{\frac{1}{2}}.e\delta,$$

accordingly as the curve is elliptic or hyperbolic.

SECTION XV.

Poles and Polars.

1. If, through a point chosen arbitrarily in the plane of a line of the second order, be drawn a series of secants to the curve; and if, through the two points of intersection of each of them with the curve, be drawn to the curve two tangents, terminated at their point of concourse: the pairs of tangents will form a series of circumscribed angles the summits of which will lie in a single straight line.

Let the equation to the conic section be

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0.$$

If $(x_{\prime}, y_{\prime})$, $(x_{\prime\prime}, y_{\prime\prime})$, be the coordinates of the two points in which any one of the secants cuts the curve, and (x', y'), those of the point of concourse of the two corresponding tangents; then, by the equations to the tangents at these two points, we shall have

$$(ax_{\prime} + cy_{\prime} + a')x' + (by_{\prime} + cx_{\prime} + b')y' + a'x_{\prime} + b'y_{\prime} + c' = 0,$$
$$(ax_{\prime\prime} + cy_{\prime\prime} + a')x' + (by_{\prime\prime} + cx_{\prime\prime} + b')y' + a'x_{\prime\prime} + b'y_{\prime\prime} + c' = 0.$$

These two equations shew that the two points $(x_{\prime}, y_{\prime})$ and $(x_{\prime\prime}, y_{\prime\prime})$ lie in a line

$$(ax + cy + a')x' + (by + cx + b')y' + a'x + b'y + c' = 0 \ldots (1),$$

where x, y, are the current coordinates.

Let h, k, be the coordinates of the point from which all the secants are drawn: since this point lies in the same line with $(x_{,}, y_{,})$ $(x_{,,}, y_{,,})$, we have

$$(ah + ck + a')\,x' + (bk + ch + b')\,y' + a'h + b'k + c' = 0 \ldots (2).$$

This equation shews that the point (x', y') always lies in a single straight line.

Cor. By virtue of the equation (2), it appears that (h, k) is always a point in the line (1). Hence we may enunciate also the following converse proposition: "If about a line of the second order be circumscribed a series of angles the summits of which all lie in a single straight line situated anywhere in the plane of the curve: the secants drawn to the curve, through the points of contact of the sides of these angles with it, all meet in a single point."

In consequence of the relation which subsists between the point (h, k), and the line

$$(ah + ck + a')\,x' + (bk + ch + b')\,y' + a'h + b'k + c' = 0,$$

the point has been called the pole of the line, and the line has been called the polar of the point.

De la Hire: *Sectiones Conicæ*, lib. II. prop. 23, 24, 26, 27.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 167, *troisième édition.*
Gergonne: *Annales de Mathématiques*, tom. III. p. 293.
Garnier: *Géométrie Analytique*, p. 161.

2. To prove that the polar is always parallel to a system of chords conjugate to the diameter passing through the pole.

The equation to a conic section which has a centre may be put under the form

$$ax^2 + by^2 + c' = 0.$$

Then, (h, k) being the pole, the equation to the polar will be, by prob. (1),

$$ahx' + bky' + c' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But the equation to any chord, conjugate to a diameter through (h, k), is, p being any constant,

$$ahx' + bky' + p = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

the equations (1) and (2) represent two parallel lines: the truth of the proposition is therefore established for central conic sections.

Again, since the equation to any conic section without a centre may be taken under the form

$$by^2 + 2a'x = 0,$$

we shall have, instead of (1) and (2), the equations

$$a'x' + bky' + a'h = 0 \ldots\ldots\ldots\ldots (3),$$

$$a'x' + bky' + p = 0 \ldots\ldots\ldots\ldots (4).$$

From these equations we see the truth of the proposition in regard to conic sections without centres.

De la Hire: *Sectiones Conicæ*, lib. II. prop. 25, 28.
Rochat: *Gergonne*, *Annales de Mathématiques*, tom. III. p. 303.

3. If two points P_1, P_2, be the poles of two lines L_1, L_2, to prove that the intersection of L_1, L_2, will be the pole of the line joining P_1, P_2.

Let (h_1, k_1) be the coordinates of P_1, and (h_2, k_2) those of P_2. Then, since P_1 is the pole of L_1, the equation to L_1 must be

$$(ah_1 + ck_1 + a')\,x + (bk_1 + ch_1 + b')\,y + a'h_1 + b'k_1 + c' = 0,$$

or $$(ax + cy + a')\,h_1 + (by + cx + b')\,k_1 + c' = 0 \ldots (1).$$

Similarly the equation to L_2 must be

$$(ax + cy + a')\,h_2 + (by + cx + b')\,k_2 + c' = 0 \ldots (2).$$

Taking now x, y, to represent the coordinates of the intersection of (1) and (2), it is plain that, x', y', representing current coordinates, the two points (h_1, k_1), (h_2, k_2), lie in a line

$$(ax + cy + a')\,x' + (by + cx + b')\,y' + c' = 0,$$

which is therefore the equation to the line P_1P_2. The form of the equation shews that P_1P_2 is the polar of the point (x, y), that is, of the intersection of L_1, L_2.

Par M. B.***, Abonné: *Gergonne*, *Annales de Mathématiques*, tom. IV. p. 379.

4. Two points P, Q, are the poles, corresponding to two polars L, M, respectively, in any conic section. To prove that, if P lie in M, Q will lie in L.

Any conic section may be represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2x}{\alpha} + \frac{2y}{\beta}.$$

Let (x_1, y_1), (x_2, y_2), be respectively the poles P, Q. Then the equations to L, M, will be, respectively,

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x + x_1}{\alpha} + \frac{y + y_1}{\beta} \ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = \frac{x + x_2}{\alpha} + \frac{y + y_2}{\beta} \ldots\ldots\ldots\ldots\ldots(2).$$

If $(x_{,}, y_{,})$, lie in (2), we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = \frac{x_1 + x_2}{\alpha} + \frac{y_1 + y_2}{\beta},$$

a result which shews that (x_2, y_2) is a point in (1).

5. To find the equation to the polar corresponding to any proposed pole, relatively to the conic section

$$uv = w^2,$$

u, v, w, being linear functions of x and y.

Let the point of contact of one of the tangents be

$$\left.\begin{matrix} \lambda u = w \\ v = \lambda w \end{matrix}\right\}:$$

then the equation to this tangent will be

$$\lambda^2 u - 2\lambda w + v = 0.$$

Let u', v', w', be the values of u, v, w, respectively, at the pole. Then

$$\lambda^2 u' - 2\lambda w' + v' = 0.$$

But, at the point of contact,

$$\lambda u = w,$$

$$v = \lambda w,$$

$$\lambda^2 u = v;$$

and therefore

$$u'v - 2w'w + v'u = 0.$$

But this equation will be true also for the contact of the other tangent. Hence, holding good for both points of contact, it will represent, when u, v, w, are regarded as variable, the equation to the chord of contact, or polar belonging to the proposed pole.

Salmon: *Treatise on Conic Sections*, p. 221.

6. If, from any point O, be drawn two straight lines OAA', OBB', intersecting a conic section in points A, A', B, B'; to prove that the intersection of AB', $A'B$, and the intersection of AB, $A'B'$, both lie in the polar belonging to O as a pole.

Let the lines OAA', OBB', be chosen as axes of x, y, respectively. Let $OA = \alpha$, $OA' = \alpha'$, $OB = \beta$, $OB' = \beta'$. Then, c being some constant quantity, the equation to the conic section will be

$$\frac{x^2}{\alpha\alpha'} + cxy + \frac{y^2}{\beta\beta'} - x\left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right) - y\left(\frac{1}{\beta} + \frac{1}{\beta'}\right) + 1 = 0.$$

The equation to the polar of O will be

$$\left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)x + \left(\frac{1}{\beta} + \frac{1}{\beta'}\right)y = 2 \ldots\ldots\ldots\ldots\ldots(1).$$

But the equations to AB, $A'B'$, AB', $A'B$, are respectively

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1, \quad \frac{x}{\alpha'} + \frac{y}{\beta'} = 1, \quad \frac{x}{\alpha} + \frac{y}{\beta'} = 1, \quad \frac{x}{\alpha'} + \frac{y}{\beta} = 1.$$

Since (1) is produced by the addition of either the first and second, or of the third and fourth of these equations, the proposition is established.

Cor. 1. Since the intersection of the tangents at A, A', and intersection of the tangents at B, B', both lie in the polar of O, it appears that the intersections of (AB', $A'B$), (AB, $A'B'$), and of the two pairs of tangents, lie all four in a single straight line.

Cor. 2. If we join two and two the reciprocal extremities of pairs of parallel chords of a conic section, the points of intersection of these new chords will all lie on the diameter bisecting the original chords.

A demonstration of this proposition in the particular case of a circle may be seen in De la Hire's *Sectiones Conicæ*, lib. I. prop. 22.

Lamé: *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*, p. 45.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 221, *troisième édition.*

Frost: *Cambridge Mathematical Journal*, vol. IV. p. 114.

7. To prove that each of the straight lines

$$u = 0, \quad v = 0, \quad w = 0,$$

is the polar of the point of intersection of the other two, relatively to the conic section

$$l^2u^2 + m^2v^2 = n^2w^2,$$

l, m, n, being any constants.

Salmon: *Treatise on Conic Sections*, p. 205.

8. To find the polar relatively to a conic section

$$uv = w^2,$$

where u, v, w, are linear functions of x and y, corresponding to a pole represented by the equations

$$\begin{Bmatrix} au = w \\ bv = w \end{Bmatrix}.$$

The equation to the required polar is

$$au - 2abw + bv = 0.$$

Salmon: *Treatise on Conic Sections*, p. 220.

9. The equation to a conic section being

$$S = l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmuv = 0,$$

to prove that the equation to the polar of a point where $u = u'$, $v = v'$, $w = w'$, is

$$u'\frac{dS}{du} + v'\frac{dS}{dv} + w'\frac{dS}{dw} = 0.$$

Salmon: *Treatise on Conic Sections*, p. 234.

10. The three sides of a triangle touch a given conic section: two of its vertices move on fixed right lines: to find the locus of the third vertex.

Let $u=0$, $v=0$, be the equations to the two tangents which pass through the intersection of the two fixed right lines. Then, the equation to the conic section will be of the form

$$uv = w^2,$$

those to the two right lines of the form

$$au = v, \quad bu = v,$$

and the equation to the locus of the third vertex will be

$$uv = \frac{4ab}{(a+b)^2} . w^2.$$

The required locus is therefore a conic section touching the proposed conic section at the ends of the polar belonging to the intersection of the two fixed lines as pole.

Salmon: *Treatise on Conic Sections*, p. 221.

11. If a point P be the pole and a line Q the corresponding polar in relation to any proposed conic section, to prove that, if P always lie in a second conic section, Q will be always a tangent to a third, and conversely, if Q be always a tangent to a second conic section, P will always lie in a third.

Brianchon: *Journal de l'Ecole Polytechnique*, *Cahier* X. p. 14.
O'Brien: *Plane Coordinate Geometry*, p. 177.

12. From a point P, without a conic section, any two straight lines are drawn each cutting the curve in two points. The points of intersection are joined, two and two, and the points, in which the joining lines (produced if necessary) cross each other, are joined by a line which will in general cut the curve in two points A, B. To prove that PA, PB, are tangents at A, B.

R. S.: *Cambridge Mathematical Journal*, vol. I. p. 32.

13. From any point O, a straight line OAC is drawn, cutting a conic section in two points A, C, and the polar of O

in B: to prove that the line OC is harmonically divided by the points A, B.

De la Hire: *Sectiones Conicæ*, lib. II. prop. 21.

14. To prove that the polars of any given point, relative to all lines of the second order which pass through any four assigned points, all meet in a single point.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 362.

15. From a pole P is drawn a straight line PS to the focus of a conic section, cutting the polar in I: the polar intersects the curve in E, F, and the directrix in G. To prove that

$$EI : FI :: EG : FG.$$

Leybourn: *Mathematical Repository, New Series*, vol. II. p. 3.

16. From a point is drawn a pair of tangents to a conic section, and a line, bisecting the interior angle between them, is drawn, cutting the corresponding polar in O: to prove that the intersection of a pair of tangents, belonging to any other polar passing through O, will lie in the line which bisects the exterior angle between the former pair of tangents.

SECTION XVI.

Polar Equations.

1. To prove that, in any conic section, if r, r', be focal distances at right angles to each other, and c the semi-latus-rectum,

$$\left(\frac{1}{r} - \frac{1}{c}\right)^2 + \left(\frac{1}{r'} - \frac{1}{c}\right)^2 = \text{a constant quantity.}$$

The equation to the conic section, θ, r, being the polar coordinates, is

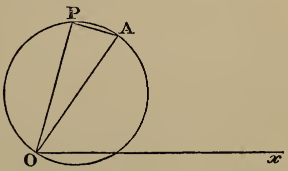
$$\frac{c}{r} = 1 + e\cos\theta:$$

putting $\frac{1}{2}\pi + \theta$, r', for θ, r, respectively, we have

$$\frac{c}{r'} = 1 - e\sin\theta:$$

hence
$$\left(\frac{c}{r}-1\right)^2+\left(\frac{c}{r'}-1\right)^2=e^2,$$
$$\left(\frac{1}{r}-\frac{1}{c}\right)^2+\left(\frac{1}{r'}-\frac{1}{c}\right)^2=\frac{e^2}{c^2}.$$

2. A line is drawn from the focus to any point of a conic section, and a circle is described upon it as a diameter: to shew that the locus of the consecutive intersections of all such circles is a circle, except, in a certain case, where it is a right line.



Let O be the focus, P the place of any point in the circle of which OA is a diameter. Let Ox, coinciding with a principal diameter of the conic section, be taken as the prime radius vector.

Let $OP=\rho$, $OA=r$, $\angle AOx=\theta$, $\angle POx=\phi$. Then the equation to the circle is
$$\rho=r\cos(\phi-\theta):$$
but the polar equation to the conic section is
$$r=\frac{c}{1+e\cos\theta},$$
where c is some constant: hence the equation to the circle becomes
$$\rho(1+e\cos\theta)=c\cos(\phi-\theta)\ldots\ldots\ldots\ldots\ldots(1).$$
Differentiating (1) with regard to θ, we have
$$e\rho\sin\theta=c\sin(\theta-\phi),$$
or
$$0=(c\cos\phi-e\rho)\sin\theta-c\sin\phi\cos\theta:$$
but, from (1), $\rho=(c\cos\phi-e\rho)\cos\theta+c\sin\phi\sin\theta.$

Squaring and adding to the last two equations, we get
$$\rho^2=(c\cos\phi-e\rho)^2+c^2\sin^2\phi$$
$$=c^2-2ce\rho\cos\phi+e^2\rho^2,$$
which is the equation to a circle unless $e=1$, in which case
$$2\rho\cos\phi=c,$$
which is the equation to a right line.

3. From any point M in the axis of a conic section, a perpendicular MPP' is drawn, cutting the curve in P: to find the locus of P' under the condition that PP' shall be always equal to the focal distance of P.

If the equation to the conic section is

$$r = \frac{c}{1 + e\cos\theta},$$

the required locus will be a conic section of the same species represented by the equation

$$r = \frac{2c\sin\theta}{1 + e\sin 2\theta}.$$

Lardner: *Algebraic Geometry*, p. 147.

4. From the focus S of a conic section are drawn n straight lines $r_1, r_2, r_3, \ldots r_n$, terminating in the curve, and dividing it into n parts which subtend equal angles at the focus: $R_1, R_2, R_3, \ldots R_n$, are straight lines drawn from the other focus to the same points. To prove that, e denoting the eccentricity of the conic section, and θ the inclination of r_1 to the greatest radius vector which can be drawn through S,

$$\frac{R_1.R_2.R_3\ldots R_n}{r_1.r_2.r_3\ldots r_n} = \frac{1 - 2e^n\cos n\theta + e^{2n}}{(1 - e^2)^n}.$$

Herschel: *Leybourn's Mathematical Repository, New Series*, p. 58.

SECTION XVII.

Linear Equation.

1. To shew that $r = fx + gy + h$ is the most general equation to a conic section referred to the focus as origin, r being the distance of the point (x, y) from the origin: also to determine the eccentricity and the position of the directrix in terms of f, g, h.

Let the equation to the directrix be

$$x \cos \alpha + y \sin \alpha = \delta,$$

and let e be the eccentricity: then, by the definition of a conic section,

$$r = e(x \cos \alpha + y \sin \alpha - \delta),$$

an equation of the required form.

Comparing this equation with the one enunciated, we have

$$f = e \cos \alpha, \quad g = e \sin \alpha, \quad h = -e\delta:$$

whence

$$e = (f^2 + g^2)^{\frac{1}{2}}, \quad \delta = -\frac{h}{(f^2 + g^2)^{\frac{1}{2}}}.$$

The equation to the directrix becomes

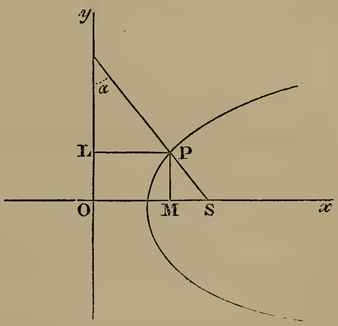
$$fx + gy + h = 0.$$

2. The focus and directrix of a conic section are given in position. Through the former a line is drawn making with the latter an angle the sine of which is equal to the eccentricity of the conic section. To find the locus of the point where this line meets the curve, the eccentricity being variable.

Let OLy be the directrix, S the focus: OSx at right angles to OLy, the axis of x, Oy being that of y. Let $OS = c$, $OM = x$, $PM = y$, $SP = r$, $e = \sin \alpha$. Then

$$r = e.PL = ex \ldots\ldots\ldots(1),$$

$$c - x = r \sin \alpha = er \ldots\ldots(2).$$



From (1) and (2),

$$r^2 = x(c - x),$$

$$(c - x)^2 + y^2 = x(c - x),$$

$$2x^2 + y^2 + c^2 = 3cx,$$

the equation to the required locus, which is therefore an ellipse.

3. Having given one of the foci of a conic section and three points in the curve, to find the equation to the directrix.

Let the focus be taken as origin of coordinates: then, $(x_{\prime}, y_{\prime})$, $(x_{\prime\prime}, y_{\prime\prime})$, $(x_{\prime\prime\prime}, y_{\prime\prime\prime})$, being the three given points, the equation to the directrix will be

$$x\{r_{,}(y_{,,}-y_{,,,})+r_{,,}(y_{,,,}-y_{,})+r_{,,,}(y_{,}-y_{,,})\}+x_{,,}y_{,,,}r_{,}+x_{,,,}y_{,}r_{,,}+x_{,}y_{,,}r_{,,,}$$
$$=y\{r_{,}(x_{,,}-x_{,,,})+r_{,,}(x_{,,,}-x_{,})+r_{,,,}(x_{,}-x_{,,})\}+y_{,,}x_{,,,}r_{,}+y_{,,,}x_{,}r_{,,}+y_{,}x_{,,}r_{,,,},$$

where $r_{,}$, $r_{,,}$, $r_{,,,}$, are used to represent respectively

$$(x_{,}^2+y_{,}^2)^{\frac{1}{2}},\quad (x_{,,}^2+y_{,,}^2)^{\frac{1}{2}},\quad (x_{,,,}^2+y_{,,,}^2)^{\frac{1}{2}}.$$

SECTION XVIII.

Polar Equation to the Tangent.

The polar equation to a conic section, when the focus is the pole, may be written in the form

$$\frac{c}{r} = 1 + e\cos\theta,$$

θ being the inclination of r to the axis containing the focus, and c denoting the latus-rectum.

The polar equation to the tangent at any point of the curve, of which α is the angular coordinate, will then be

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha).$$

This expression for the polar equation to the tangent of a conic section is due to Mr. Davies, by whom it was communicated to the *Philosophical Magazine* for 1842, p. 192.

1. If S be the focus of a conic section, and T the intersection of the tangents at the extremities P, p, of any focal chord PSp; to prove that the line ST, joining S and T, is at right angles to PSp.

Let α, $\pi+\alpha$, be the angular coordinates of P, p, respectively: then the equations to PT, pT, will be respectively

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha),$$

$$\frac{c}{r} = e\cos\theta - \cos(\theta - \alpha).$$

Hence, at the intersection T of these two lines,

$$\cos(\theta - \alpha) = 0,$$

$$\theta - \alpha = \pm \tfrac{1}{2}\pi,$$

$$\theta = \alpha \pm \tfrac{1}{2}\pi,$$

which establishes the proposition.

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 23.

2. To find the angle subtended at the focus by the tangent drawn to a central conic section from any assigned point.

The polar equation to the conic section being

$$\frac{c}{r} = 1 + e\cos\theta,$$

that to the tangent at a point, of which the angular coordinate is α, will be

$$\frac{c}{r} = e\cos\theta + \cos(\alpha - \theta).$$

But, x being the abscissa, on the major axis, of any point r, θ, in the tangent,

$$\pm(x - ae) = r\cos\theta,$$

$$\frac{c}{r} = \pm(x - ae)\frac{e}{r} + \cos(\alpha - \theta),$$

and

$$c = \pm a(1 - e^2),$$

the + or − sign being taken as the curve is the ellipse or the hyperbola: hence

$$\pm\frac{a - ex}{r} = \cos(\alpha - \theta):$$

which determines the required angle $\alpha - \theta$ in terms of x and r.

O'Brien: *Plane Coordinate Geometry*, p. 156.

3. The angle subtended, at one of the foci of a conic section, by the part of a moveable tangent intercepted between two fixed tangents is always constant.

The polar equation to the conic section, when the focus is the pole, being of the form

$$\frac{c}{r} = 1 + e\cos\theta,$$

the polar equations to the moveable and fixed tangents will be

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha)\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha')\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha'')\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

α, α', α'', being the values of θ at the three points of contact respectively.

At the intersection of (1) and (2),

$$\cos(\theta - \alpha) = \cos(\theta - \alpha'),$$

whence $$\theta = \tfrac{1}{2}(\alpha' + \alpha).$$

Similarly, at the intersection of (1) and (3),

$$\theta = \tfrac{1}{2}(\alpha'' + \alpha).$$

The angle, subtended by the intercepted portion of the tangent (1) at the focus, will be equal to the difference between these two values of θ, that is, to $\frac{1}{2}(\alpha' \sim \alpha'')$, a constant quantity.

Poncelet: *Gergonne, Annales de Mathématiques*, tom. VIII. p. 4.

4. Two conic sections have a common focus, through which any radius vector is drawn, meeting the curves in P, P', respectively. To find the locus of the point of intersection of the tangents at P, P'.

The equations to the conic sections are of the forms

$$\frac{c}{r} = 1 + e\cos\theta,$$

$$\frac{c'}{r} = 1 + e'\cos(\theta + \varepsilon),$$

and the equations to the tangents at P, P', the angular coordinate of both points being α, will be

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{c'}{r} = e'\cos(\theta + \varepsilon) + \cos(\theta - \alpha)\ldots\ldots\ldots\ldots(2).$$

At the intersection of (1) and (2),

$$\frac{c-c'}{r} = e\cos\theta - e'\cos(\theta+\varepsilon)\ldots\ldots\ldots\ldots\ldots(3),$$

the equation to the required locus, which is therefore a straight line.

Cor. 1. The equations to the directors of the two conic sections are

$$\frac{c}{r} = e\cos\theta,$$

and

$$\frac{c'}{r} = e'\cos(\theta+\varepsilon).$$

Hence the line (3) passes through the intersection of the two directors.

Cor. 2. If l, l', denote the sines of the angles which (3) makes with the two directors, it may easily be shewn that

$$\frac{l}{l'} = \frac{e'}{e}.$$

5. To circumscribe a triangle about a given conic section, such that each of its summits shall lie in a given line.

Let the equations to the three given lines be

$$\left.\begin{aligned}\delta_1 &= r\cos(\theta-\varepsilon_1)\\ \delta_2 &= r\cos(\theta-\varepsilon_2)\\ \delta_3 &= r\cos(\theta-\varepsilon_3)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equations to the three sides of the circumscribing triangle may be written in the form

$$\left.\begin{aligned}\frac{c}{r} &= e\cos\theta + \cos(\theta-\alpha_1)\\ \frac{c}{r} &= e\cos\theta + \cos(\theta-\alpha_2)\\ \frac{c}{r} &= e\cos\theta + \cos(\theta-\alpha_3)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(2).$$

Conceive the first and second of the lines (2) to intersect in the last of the lines (1).

At the intersection of the first and second of the lines (2), we have

$$\theta = \tfrac{1}{2}(\alpha_1 + \alpha_2),$$

$$\frac{c}{r} = e\cos\frac{\alpha_1 + \alpha_2}{2} + \cos\frac{\alpha_1 - \alpha_2}{2},$$

and therefore, by the last of the equations (1),

$$\delta_3\left(e\cos\frac{\alpha_1 + \alpha_2}{2} + \cos\frac{\alpha_1 - \alpha_2}{2}\right) = c\cos\left(\frac{\alpha_1 + \alpha_2}{2} - \varepsilon_3\right),$$

whence, putting $\tan\frac{1}{2}\alpha_1 = u_1$ and $\tan\frac{1}{2}\alpha_2 = u_2$, we have

$$\delta_3\{1 + e + (1 - e)u_1u_2\} = c\cos\varepsilon_3 - cu_1u_2\cos\varepsilon_3 + c\sin\varepsilon_3(u_1 + u_2),$$

$$\frac{c\cos\varepsilon_3 + (1 - e)\delta_3}{c\sin\varepsilon_3}.u_1u_2 = u_1 + u_2 + \cot\varepsilon_3 - \frac{(1 + e)\delta_3}{c\sin\varepsilon_3},$$

or, a_3, b_3, denoting known quantities,

$$a_3u_1u_2 = u_1 + u_2 + b_3.$$

Similarly, in relation to the other two angular points of the triangle, we have

$$a_1u_2u_3 = u_2 + u_3 + b_1,$$

$$a_2u_3u_1 = u_3 + u_1 + b_2.$$

Eliminating u_2 and u_3 between these three equations we shall get a quadratic in u_1, and thus u_1, u_2, u_3, may be determined. The double value of u_1 shews that generally there will be two solutions of the problem.

A solution of this problem, in the particular case of the circle, may be seen in Mr. Hearn's excellent work entitled *Researches on Curves of the Second Order*, p. 17.

6. To find the distance of the pole from a tangent to a conic section

$$r = \frac{c}{1 + e\cos\theta},$$

drawn at a point where $\theta = \alpha$.

If p denote the required distance,

$$p^2 = \frac{c^2}{1 + e^2 + 2e\cos\alpha}.$$

7. The tangents at the extremities of the major axis of a conic section intersect any other tangent in two points: to prove that the straight lines joining these two points with either focus, are at right angles to each other.

'ΑΠΟΛΛΩΝΙΟΥ ΠΕΡΓΑΙΟΥ Κωνικῶν τὸ τρίτον· Πρότασις με'.

8. If, from a focus of a conic section, lines be drawn to the summit of the angle formed by any two tangents and to the two points of contact, the first of these three lines will bisect the angle formed by the two others.

De la Hire: *Sectiones Conicæ*, lib. VIII. prop. 24.
Poncelet: *Gergonne*, *Annales de Mathématiques*, tom. VIII. p. 5.
De Morgan: *Cambridge Mathematical Journal*, vol. II. p. 202.

9. When a moveable tangent to a conic section terminates in two fixed tangents, the sum of the angles which the first tangent subtends at the two foci, in the ellipse, and the difference, in the hyperbola, is constant and equal to the supplement of the angle between the two fixed tangents.

Poncelet: *Gergonne*, *Annales de Mathématiques*, tom. VIII. p. 8.

10. To find the locus of the intersection of a tangent to a conic section with a straight line drawn through the focus, at right angles to the radius vector of the point of contact.

The locus will be the directrix.

Chasles: *Gergonne*, *Annales de Mathématiques*, tom. XVIII. p. 273.

11. To prove that two confocal conic sections cannot have more than two common tangents.

Chasles: *Gergonne*, *Annales de Mathématiques*, tom. XVIII. p. 273.

12. If a variable angle, circumscribed about a conic section, moves in such a manner as to intercept between its sides, a portion of any fixed tangent subtending a constant angle at the focus, to find the locus of the summit of the variable angle.

The equation to the conic section being

$$\frac{c}{r} = 1 + e\cos\theta,$$

the required locus will be a confocal conic section, having the same directrix as the original one, and defined by the equation

$$\frac{c}{r} = \cos\varepsilon + e\cos\theta,$$

ε being the constant angle.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 190.

13. The angle between the focal distances of two points of a conic section being given, to find the locus of the intersection of the tangents at these points.

Let α denote the given angle; then, the equation to the conic section being

$$\frac{c}{r} = 1 + e\cos\theta,$$

the required locus will be a conic section defined by the equation

$$\frac{c}{r} = \cos\alpha + e\cos\theta.$$

14. The focal distances of two points of a conic section include a given angle: to find the focal distance of the intersection of the tangents at these points.

If 2α denote the given angle between the focal distances r, r', and the equation to the conic section be

$$\frac{c}{r} = 1 + e\cos\theta,$$

the square of the required distance is equal to

$$\frac{c^2rr'}{c^2 - (1 - e^2)\,rr'.\sin^2\alpha}.$$

SECTION XIX.

Polar Equation to the Chord of a Conic Section.

If the equation to the conic section be

$$\frac{c}{r} = 1 + e\cos\theta,$$

the equation to the chord through points, the angular coordinates of which are $\alpha + \beta$ and $\alpha - \beta$, will be

$$\frac{c}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta.$$

This very useful form of the equation to the chord of a conic section was inserted by Mr. Frost in the *Cambridge and Dublin Mathematical Journal*, vol. I. p. 68.

1. PSP', QSQ', are two focal chords of a conic section: to prove that a line through S, bisecting the angle PSQ', will intersect the chord QP, produced indefinitely, in the directrix.

The equation to the conic section being

$$\frac{c}{r} = 1 + e\cos\theta,$$

that to the chord QP will be, $\alpha - \beta$, $\alpha + \beta$, being the angular coordinates of P, Q, respectively,

$$\frac{c}{r} = \sec\beta.\cos(\theta - \alpha) + e\cos\theta \ldots\ldots\ldots\ldots\ldots(1).$$

But the angular coordinate of the line bisecting the angle PSQ' is equal to

$$\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} - \frac{\pi}{2}$$

$$= \alpha - \frac{\pi}{2},$$

and therefore, at its point of intersection with (1),

$$\frac{c}{r} = e\cos\theta,$$

a result which shews that the intersection lies in the directrix.

Cor. If we have given any three points in a conic section and the focus, we can easily construct the directrix, two of its points being readily determinable by the aid of this proposition.

Chasles: *Gergonne*, *Annales de Mathématiques*, tom. XVIII. p. 275.

2. If a variable angle, circumscribed about a conic section, moves in such a manner as to intercept, between its sides, a portion of any fixed tangent subtending a constant angle at the focus, to find the equation to the chord joining the points of contact of the two sides of the variable angle.

Let the equation to the fixed tangent be

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha)\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and those to the sides of the variable angle,

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \lambda)\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \mu)\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

At the intersection of (1) and (2), $2\theta = \alpha + \lambda$, and, at the intersection of (1) and (3), $2\theta = \alpha + \mu$: hence, ε denoting the constant difference in the values of θ at the two intersections,

$$2\varepsilon = \lambda - \mu\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation to the chord through the points of contact of (2) and (3), the angular coordinates of which are λ, μ, will be

$$\frac{c}{r} = \sec\frac{\lambda - \mu}{2}.\cos\left(\theta - \frac{\lambda + \mu}{2}\right) + e\cos\theta,$$

and therefore, by (4), we have for the required equation to the chord,

$$\frac{c\cos\varepsilon}{r} = e\cos\varepsilon\,\cos\theta + \cos\left(\theta - \frac{\lambda + \mu}{2}\right).$$

Cor. This result shews that the chord of contact is always a tangent to a confocal conic section which has the same directrix as the original one.

Bobillier: *Gergonne*, *Annales de Mathématiques*, tom. XVIII. p. 190.

3. Two focal chords of a conic section are drawn and lines joining their extremities are produced to meet in two points P, Q: to prove that, S being the focus, SP is at right angles to SQ.

Let ESF, $E'SF'$, be the two focal chords, the angular coordinates of E, E', being $\alpha - \beta$, $\alpha + \beta$, and, consequently, those of F, F', $\pi + \alpha - \beta$, $\pi + \alpha + \beta$.

The equation to EE' is

$$\frac{c}{r} = \sec\beta.\cos(\theta - \alpha) + e\cos\theta,$$

and to FF', $\pi + \alpha$ being substituted for α,

$$\frac{c}{r} = -\sec\beta\cos(\theta - \alpha) + e\cos\theta.$$

Hence, at P, the intersection of EE', FF',

$$\cos(\theta - \alpha) = 0 \text{..........................(1)},$$

and therefore $$\theta = \alpha + (2\lambda + 1)\tfrac{1}{2}\pi,$$

λ being any integer, that is, the angular coordinate of P is equal to half the sum of the angular coordinates of E, E', or F, F', together with an odd multiple of $\tfrac{1}{2}\pi$.

By similarity of circumstances therefore, $\theta_{,}$ denoting the angular coordinate of Q,

$$\theta_{,} = \tfrac{1}{2}\pi + \alpha + (2\mu + 1)\tfrac{1}{2}\pi = \alpha + (\mu + 1)\pi,$$

μ being any integer.

Hence $$\theta_{,} - \theta = (\mu - \lambda)\pi + \tfrac{1}{2}\pi,$$

which shews that SP is at right angles to SQ.

4. A chord PQ of a conic section passes always through a given point C: supposing PS, QS, CS, to be joined, S being a focus of the curve, to prove that

$$\tan\frac{PSC}{2}.\tan\frac{QSC}{2} = \text{a constant quantity.}$$

The angular coordinates of P, Q, being $\alpha - \beta$, $\alpha + \beta$, the equation to PQ will be of the form

$$\frac{c}{r} = \sec\beta.\cos(\theta - \alpha) + e\cos\theta.$$

Let $r_{,}$, $\theta_{,}$, be the coordinates of C: then

$$\frac{c}{r_{,}} = \sec\beta.\cos(\theta_{,} - \alpha) + e\cos\theta_{,} \ldots\ldots\ldots\ldots\text{(1)}.$$

Again, $$\tan\frac{PSC}{2}.\tan\frac{QSC}{2} = \tan\frac{\theta_{,} - \alpha + \beta}{2}.\tan\frac{\alpha + \beta - \theta_{,}}{2}$$

$$= \frac{\cos(\theta_{,} - \alpha) - \cos\beta}{\cos(\theta_{,} - \alpha) + \cos\beta}.$$

But, from (1), $\cos(\theta_{,} - \alpha) = \cos\beta.\left(\frac{c}{r_{,}} - e\cos\theta_{,}\right)$.

Hence $$\tan\frac{PSC}{2}.\tan\frac{QSC}{2} = \frac{\frac{c}{r_{,}} - e\cos\theta_{,} - 1}{\frac{c}{r_{,}} - e\cos\theta_{,} + 1},$$

which is a constant quantity.

Stubbs: *Annals of Philosophy*, for November, 1843, p. 341.

5. To inscribe a triangle in a given conic section, so that its sides may pass respectively through three given points.

Let the equation to the conic section be

$$\frac{c}{r} = 1 + e\cos\theta:$$

and let α_1, α_2, α_3, be the angular coordinates of the summits of the inscribed triangle. The equation to the chord joining the first two of these summits is

$$\left(\frac{c}{r} - e\cos\theta\right)\cos\frac{\alpha_1 - \alpha_2}{2} = \cos\left(\theta - \frac{\alpha_1 + \alpha_2}{2}\right).$$

Let r_3, θ_3, be the coordinates of the given point through which this chord passes: then, expanding the cosines and putting $\tan\frac{1}{2}\alpha_1 = u_1$, $\tan\frac{1}{2}\alpha_2 = u_2$, we have

$$\left(\frac{c}{r_3} - e\cos\theta_3\right).(1 + u_1u_2) = \cos\theta_3.(1 - u_1u_2) + \sin\theta_3.(u_1 + u_2),$$

or, a_3, b_3, representing known quantities,

$$a_3u_1u_2 = u_1 + u_2 + b_3 \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}.$$

Similarly, by considering the two other chords we have

$$a_1u_2u_3 = u_2 + u_3 + b_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$a_2u_3u_1 = u_3 + u_1 + b_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$[*]

From (1) and (3) we see that

$$u_2 = \frac{u_1 + b_3}{a_3u_1 - 1}, \qquad u_3 = \frac{u_1 + b_2}{a_2u_1 - 1},$$

and therefore, from (2),

$$a_1(u_1 + b_2)(u_1 + b_3) = (a_2u_1 - 1)(u_1 + b_3) + (a_3u_1 - 1)(u_1 + b_2) \\ + b_1(a_2u_1 - 1)(a_3u_1 - 1),$$

and consequently

$$(a_1 - a_2 - a_3 - a_2a_3b_1)u_1^2 + \{2 + a_1(b_2 + b_3) + b_1(a_2 + a_3) - a_2b_3 - a_3b_2\}u_1 \\ - b_1 + b_2 + b_3 + a_1b_2b_3 = 0.$$

From this equation may be determined two values for u_1, the corresponding values of u_2 and u_3 being then given by (2) and (3): thus α_1, α_2, α_3, are ascertained. The double value of u_1 shews that there are two triangles which will satisfy the conditions of the problem.

The earliest solutions of this problem, in the particular case of the circle, are due to De Castillon and Lagrange, the former being geometrical and the latter trigonometrical. The following is Castillon's history of the problem.

"Je vais parler d'un Problême que je n'ai pas imaginé, et que pourtant, dans sa généralité, je n'ai vu proposé nulle part. Je dis *dans sa généralité*, parce que Pappus dans la Propos. 117, Probl. 40 de sa Collection, en propose un des cas les plus faciles, et en donne une solution qui ne peut pas s'appliquer aux autres cas. Selon mes conjectures, quelque Amateur de la Géométrie des Anciens généralisa le Problême de Pappus, et, ne l'ayant résolu qu' avec beaucoup de peine, ou, peut-être, ne l'ayant point résolu, il le proposa verbalement à quelqu' un qui couroit la même carriere. Celui-ci suivit l'exemple de son Prédécesseur; et de main en main le Problême est parvenu

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[*] The formulæ (1), (2), (3), coincide in form with those obtained by Lagrange in the particular case of a circle.

jusqu' à moi. Il semble que le petit nombre de Géométres qui le connoissoient, le gardoient pour embarrasser les autres dans les occasions. Je n'en sais de certain que ce qui me regarde: en voici l'histoire.

"Feu Mr. Cramer, célèbre Professeur à Geneve, et digne Membre externe de cet illustre Corps, m'honoroit de son amitié. Il s'apperçut que j'aimois beaucoup la méthode Géométrique des Anciens. Il en prit occasion en 1742 de me proposer le Problême en question, qu' il falloit résoudre par l'ancienne Analyse. 'Dans ma jeunesse, me dit-il, j'avois le goût que vous avez: un vieux Géométre, pour essayer mes forces en ce genre, me proposa le Problême que je vous propose: tentez de le résoudre; et vous verrez combien il est difficile.'

"Je l'entrepris: je trouvai quelques Théorèmes, qui, à ce que je croyois, m'approchoient du but, mais qui ne purent m'y faire atteindre. Dans la suite de ce Mémoire on en trouvera quelques-uns qui viennent à propos. Peu de temps après m'avoir parlé de ce Problême, Mr. Cramer par ses exhortations m'engagea à donner les Opuscules de Newton; et les soins que cette édition demandoit, me firent tellement oublier le Problême dont il s'agit, que je n'y songeai plus.

"On m'en fit souvenir en 1755. Un des amis que j'avois à la Haye, étoit Mr. Bouquet, à qui je rendis justice dans le Mémoire *sur le terme général des séries recurrentes*, que je pris la liberté de soumettre au jugement de cette Compagnie, lorsqu' elle me fit l'honneur de me mettre au nombre de ses Membres externes. Depuis j'ai eu le plaisir de voir la bonne opinion que j'avois de mon ami en qualité d'homme de lettres, confirmée par l'illustre Franklin, et le cas que j'en fesois comme militaire, justifié par les éloges de toute l'Angleterre.

"Mr. Bouquet m'écrivit donc qu' un Anonyme avoit proposé à la Haye le Problême dont il s'agit, comme digne de l'attention des Géométres. Mr. Bouquet ajoûta qu' il ne doutoit pas de mon courage à l'attaquer, et de mon bonheur à m'en rendre maître. Les leçons que je devois donner à Utrecht, me laissoient très-peu de temps. Pendant quelques semaines, je l'employai à cette recherche; mais sans fruit. L'inutilité de mes

efforts me piqua sans me décourager. Comme personne ne déclaroit publiquement avoir résolu ce Problême, je continuai à y travailler, mais seulement quand j'étois de loisir. Je fis bien des tentatives inutiles, je l'avoue. Enfin il me vint dans l'esprit qu' un Théorême de Pappus s'appliquoit naturellement à la figure que j'avois construite. Je tentai cette application; et j'aperçus d'abord qu'elle me donnoit la solution désirée. C'est ainsi que j'y parvins; mais si tard qu' on ne parloit plus ni de l'Anonyme, ni de son Problême. Je mis donc ma solution parmi mes papiers, où je viens de la retrouver. J'ai cru que je ferois bien de la publier, pour épargner aux Géométres à venir la peine et la perte du temps que ce Problême pourroit leur coûter."

De Castillon: *Mémoires de l'Académie des Sciences et Belles-Lettres, Berlin*, 1776, p. 265, (geometrically for the circle).

Lagrange: *Ibid.*, p. 284, (trigonometrically for the circle).

Euler: *Acta Academiæ Scientiarum Petropolitanæ*, 1780, *pars prior*, p. 91, (for the circle).

Fuss: *Ibid.*, *pars prior*, p. 97, (for the circle).

Lexell: *Ibid.*, *pars posterior*, p. 70, (construction of Lagrange's formula for the circle).

Oltajano*: *Mémorie della Societa Italiana*, tom. IV. p. 4, 1788, (synthetically for any polygon inscribed in a circle).

Malfatti: *Ibid.*, tom. IV. p. 201, (synthetically for any polygon inscribed in a circle).

Lhuilier: *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, p. 277, &c., (geometrically, for a triangle, or any polygon inscribed in a circle).

Gergonne: *Annales de Mathématiques*, 1816, 1817, tom. VII. p. 325, (algebraically for the parabola).

Carnot: *Géométrie de position*, p. 384, (algebraically, for the circle, by a method applicable to an inscribed polygon of any number of sides).

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* Oltajano's solution was presented to the Societa Italiana when he was about sixteen years of age.

Lamé: *Examen de différentes méthodes émployées pour résoudre les problémes de Géométrie*, p. 58, (algebraically, for any conic section).

Puissant: *Recueil de diverses propositions de Géométrie*, p. 125, (algebraically, for the circle, by a method applicable to any conic section).

Hearn: *Researches on Curves of the Second Order*, (algebraically, for the circle).

Gaskin: *Solutions of Geometrical Problems*, p. 166, (algebraically, for each of the conic sections).

Salmon: *Treatise on Conic Sections*, p. 222.

6. If the chord of a conic section, the eccentricity of which is e, subtends at its focus a constant angle 2α, to prove that it always touches a conic section, having the same focus, and of which the eccentricity is $e\cos\alpha$.

Booth: *Cambridge Mathematical Journal*, vol. III. p. 87.

Ellis: *Ibid.*, p. 94.

Frost: *Cambridge and Dublin Mathematical Journal*, vol. I. p. 69.

7. Chords are drawn in a conic section, so as to subtend a constant angle at the focus: to find the locus of the intersection of each such chord with the perpendicular upon it drawn from the focus.

The equation to the conic section being

$$\frac{c}{r} = 1 + e\cos\theta,$$

that to the required locus will be

$$c.(c - 2er\cos\theta) = (1 - e^2\cos^2\beta).r^2,$$

which is the equation to a circle, unless $e\cos\beta = 1$, when it becomes that of a straight line.

8. If the angle between two focal distances be bisected by a third, which remains fixed in position, to prove that the chords joining the extremities of the two focal distances, as they change their position, always pass through a fixed point in the directrix.

Frost: *Cambridge and Dublin Mathematical Journal*, vol. I. p. 69.

9. The angular coordinates of the extremities of two chords in a conic section being $\alpha + \beta$, $\alpha - \beta$, and $\alpha' + \beta'$, $\alpha' - \beta'$, to find the locus of their intersection, having given that

$$\beta' = \beta, \quad \text{and} \quad \alpha' - \alpha = \varepsilon,$$

where ε is a constant.

The required locus will be a conic section defined by the equation

$$\frac{c}{r} = \sec\beta . \cos\frac{\varepsilon}{2} + e\cos\theta.$$

Frost: *Cambridge and Dublin Mathematical Journal*, vol. I. p. 70.

10. A chord being inscribed arbitrarily in a conic section, if, from one of its foci, be drawn radii vectores (1) and (2) to the two extremities of this chord, a radius vector (3) to the point where this chord cuts the directrix belonging to this focus, a radius vector (4) to the summit of the circumscribed angle which touches the curve at the extremities of the inscribed chord, and two other radii vectores (5) and (6) to the points where the sides of this circumscribed angle cut this same directrix; then the radius vector (4) will bisect the angle between the two radii vectores (1) and (2); the radius vector (3) will bisect the angle between the two radii vectores (5) and (6), and the two radii vectores (3) and (4) will be at right angles to each other.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 190.

SECTION XX.

Inscribed Polygons.

1. If from any point in a conic section perpendiculars be drawn to the sides of a quadrilateral inscribed in it, to prove that the product of the perpendiculars on one pair of opposite sides is to the product of the perpendiculars on the other pair of sides in a constant ratio.

Let the equations to the four sides of the quadrilateral taken in order be

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad u_4 = 0,$$

the general type of the equation to a straight line being taken in the form

$$x \cos\alpha + y \sin\alpha - \delta = 0.$$

Then the equation to the conic section will be of the form

$$\lambda u_1 u_3 + \mu u_2 u_4 = 0,^*$$

λ and μ being constants.

But $u_1 u_3$ is equal to the product of the perpendiculars from any point (x, y) in the curve upon the sides $u_1 = 0$, $u_3 = 0$: and $u_2 u_3$ to that of the perpendiculars on the other two sides.

Hence the ratio of the two products is equal to a constant quantity $-\dfrac{\mu}{\lambda}$.

Sturm: *Gergonne*, *Annales de Mathématiques*, tom. XVII. p. 179.

2. If three sides of a quadrilateral, inscribed in a given conic section, pass always through three given points of a given straight line, to prove that the fourth side also will always pass through a given point of the same line.

Let the given straight line be taken as the axis of y, and let the axis of x be a diameter conjugate to chords parallel to the given line. Then the equation to the conic section will be

$$y^2 = l + mx + nx^2 \text{.....................} (1).$$

Let the equations to the four sides, taken in order, be

$$y = ax + b, \quad y = a'x + b', \quad y = \alpha x + \beta, \quad y = \alpha' x + \beta'.$$

Then, ρ being a constant multiplier, the equation to the conic section must also be

$$(y - ax - b)(y - \alpha x - \beta) = \rho (y - a'x - b')(y - \alpha' x - \beta') \text{...} (2).$$

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[*] This expression for the equation to a conic section circumscribing a quadrilateral was given by Bobillier in the *Annales de Mathématiques* of Gergonne.

Since (1) and (2) must be coincident, we have, by comparing the constant terms and the coefficients of y,

$$\rho b'\beta' - b\beta = (1 - \rho)\, l,$$

and

$$b + \beta = \rho\,(b' + \beta');$$

whence, eliminating ρ, we have,

$$l\,(b + \beta - b' - \beta') = b\beta\,(b' + \beta') - b'\beta'\,(b + \beta) \ldots\ldots (3).$$

Now, by the conditions of the problem, three of the quantities b, b', β, β', are known; hence, by (3), the fourth is also determined, and the theorem is established.

O'Brien: *Plane Coordinate Geometry*, p. 178.

3. If two consecutive sides of a hexagon, inscribed in a conic section, are respectively parallel to the sides opposite to them, the other two sides of the hexagon will also be parallel to each other.

Let the two sides A, B, of a hexagon $ABCDEF$, inscribed in a conic section, be parallel respectively to the two sides D, E. Then, if we take the sides A, B, as the axes of y, x, respectively, the equations to A, B, D, E, will be respectively of the forms

$$x = 0, \quad y = 0, \quad x + a = 0, \quad y + b = 0.$$

Let the equations to the sides C, F, and to the diagonal through the points (A, B), (D, E), be respectively

$$u = 0, \quad v = 0, \quad w = 0.$$

Then, since the conic section circumscribes each of the quadrilaterals into which the diagonal divides the hexagon, the equation to the conic section must be indifferently of either of the following forms,

$$uw = \lambda y\,(x + a),$$

$$vw = \mu x\,(y + b);$$

where λ, μ, are constants.

Since these two equations must be identical, it follows that the ratio of the coefficients of x^2 to that of y^2 must be the same in both, and therefore in uw, vw; hence the ratio of the coefficient of x to that of y must be the same in u and v. This conclusion shews that C is parallel to F.

Gergonne: *Annales de Mathématiques*, tom. IV. p. 79.

4. A triangle is inscribed in a conic section: to prove that the points, in which the sides of the triangle produced meet the tangents at the opposite angles, are in the same straight line.

Let the equations to the sides of the inscribed triangle be

$$u = 0 \ldots\ldots\ldots (1), \quad v = 0 \ldots\ldots\ldots (2), \quad w = 0 \ldots\ldots\ldots (3).$$

The equation to the conic section will then be

$$\frac{\lambda}{u} + \frac{\mu}{v} + \frac{\nu}{w} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).^*$$

Now it is easily seen that the line denoted by

$$\frac{v}{\mu} + \frac{w}{\nu} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

is the tangent to (4) at the intersection of (2) and (3); for the equations (4) and (5) combined are equivalent to (2) and (3) taken simultaneously.

Again (1) and (5) evidently intersect in a line

$$\frac{u}{\lambda} + \frac{v}{\mu} + \frac{w}{\nu} = 0.$$

Symmetry shews that this must be a line in which all the three intersections, mentioned in the problem, take place.

Carnot: *Géométrie de Position*, p. 453.

Salmon: *Treatise on Conic Sections*, p. 235.

5. $AA'B'B$ is a quadrilateral inscribed in a conic section, the sides $A'A$, $B'B$, produced, intersecting in O. If a straight line $OPHH'P'$ be drawn through O, cutting the conic section in P, P', and the quadrilateral in H, H': to prove that

$$\frac{1}{OP} + \frac{1}{OP'} = \frac{1}{OH} + \frac{1}{OH'}.$$

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* Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 320.

Let OAA', OBB', be taken as axes of x, y, respectively.

Let $OA = \alpha$, $OA' = \alpha'$, $OB = \beta$, $OB' = \beta'$. Then, c being any constant, the equation to the conic section is

$$\frac{x^2}{\alpha\alpha'} + cxy + \frac{y^2}{\beta\beta'} - \frac{x}{\alpha\alpha'}(\alpha + \alpha') - \frac{y}{\beta\beta'}(\beta + \beta') + 1 = 0.$$

The equation to AB is $\frac{x}{\alpha} + \frac{y}{\beta} = 1,$

and to $A'B'$ $\frac{x}{\alpha'} + \frac{y}{\beta'} = 1.$

Let the equation to $OPHH'P'$ be

$$y = mx.$$

Then, x being the abscissa of H,

$$\frac{1}{x} = \frac{1}{\alpha} + \frac{m}{\beta}.$$

Similarly, x' being the abscissa of H',

$$\frac{1}{x'} = \frac{1}{\alpha'} + \frac{m}{\beta'},$$

and therefore $\frac{1}{x} + \frac{1}{x'} = \frac{1}{\alpha} + \frac{1}{\alpha'} + m\left(\frac{1}{\beta} + \frac{1}{\beta'}\right).$

At the points P and P' we have

$$\frac{1}{x^2} - \left\{\frac{1}{\alpha\alpha'}(\alpha + \alpha') + \frac{m}{\beta\beta'}(\beta + \beta')\right\}\frac{1}{x} + \frac{1}{\alpha\alpha'} + cm + \frac{m^2}{\beta\beta'} = 0.$$

Hence, X, X', being the abscissæ of P, P',

$$\frac{1}{X} + \frac{1}{X'} = \frac{1}{\alpha} + \frac{1}{\alpha'} + m\left(\frac{1}{\beta} + \frac{1}{\beta'}\right).$$

Consequently $\frac{1}{X} + \frac{1}{X'} = \frac{1}{x} + \frac{1}{x'}.$

But it is evident, from the geometry, that X, X', x, x', are proportional to OP, OP', OH, OH': hence

$$\frac{1}{OP} + \frac{1}{OP'} = \frac{1}{OH} + \frac{1}{OH'}.$$

6. The opposite sides of any hexagon, inscribed in a conic section, intersect in three points lying in a straight line.

Let *ABCDEF* be a hexagon inscribed in a conic section, and let the equations to the alternate sides *AB*, *CD*, *EF*, be respectively, when multiplied by arbitrary constants,

$$u = 0 \ldots\ldots\ldots (1); \qquad v = 0 \ldots\ldots\ldots (2); \qquad w = 0 \ldots\ldots\ldots (3).$$

The equation to the conic section may be expressed under the form

$$u^2 + v^2 + w^2 - (\lambda + \lambda^{-1})\, vw - (\mu + \mu^{-1})\, wu - (\nu + \nu^{-1}) uv = 0 \ldots (4);$$

for this is an equation of the second order, involving five arbitrary constants, which depend upon the three quantities $\lambda + \lambda^{-1}$, $\mu + \mu^{-1}$, $\nu + \nu^{-1}$, and the two ratios between the three arbitrary constant factors included in u, v, w.

To determine the points A and B, put $u = 0$: then

$$v^2 + w^2 - (\lambda + \lambda^{-1})\, vw = 0,$$

whence $$v = \lambda w \quad \text{or} \quad w = \lambda v;$$

and, in like manner, by putting successively $v = 0$ and $w = 0$ in (4), we shall get the equations which determine the positions of the other angular points of the hexagon. The equations to the points A, B, C, D, E, F, are respectively the following:

$$A \ldots\ldots \left\{\begin{array}{l} u = 0 \\ v = \lambda w \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$B \ldots\ldots \left\{\begin{array}{l} u = 0 \\ w = \lambda v \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$C \ldots\ldots \left\{\begin{array}{l} v = 0 \\ w = \mu u \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$D \ldots\ldots \left\{\begin{array}{l} v = 0 \\ u = \mu w \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$E \ldots\ldots \left\{\begin{array}{l} w = 0 \\ u = \nu v \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$F \ldots\ldots \left\{\begin{array}{l} w = 0 \\ v = \nu u \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Hence (8, 9), (10, 5), (6, 7), the equations to the lines *DE*, *FA*, *BC*, respectively, are, by mere inspection, observed to be

$$u = \mu w + \nu v \qquad \text{(11)},$$

$$v = \nu u + \lambda w \qquad \text{(12)},$$

$$w = \lambda v + \mu u \qquad \text{(13)}.$$

Now each of the three pairs of equations (1, 11), (2, 12), (3, 13), evidently satisfies the equation

$$\frac{u}{\lambda} + \frac{v}{\mu} + \frac{w}{\nu} = 0 \qquad \text{(14)};$$

and therefore the opposite sides of the hexagon *ABCDEF* intersect in three points in a straight line of which (14) is the equation.

The demonstration here given of the property of the inscribed hexagon, is due to Mr. Weddle, by whom it was communicated to the *Cambridge and Dublin Mathematical Journal*, vol. III. p. 285.

This beautiful property was discovered by Pascal,* and formed the basis of a complete system of conic sections, written at the age of sixteen, which was never published and of which the manuscript was unfortunately lost. In this treatise he is said by Mersenne† to have demonstrated all the propositions of the Conics of Apollonius, emanating in 400 corollaries from this one fundamental theorem.‡ This work having been sent to Descartes, he could not be prevailed upon to believe it to have been written by one so young, ascribing it either to Pascal's father or to his friend Desargues. The early genius however of Pascal, exhibited in other matters of like nature, renders his title to these discoveries in no degree improbable. Leibnitz,‖ in allusion to Pascal, says, "Mr. Périer, son neveu, me donna un jour à lire et à ranger un excellent ouvrage de son oncle sur les coniques, et j'espérois qu' on le publieroit d'abord. On lui auroit conservé par-là l'honneur d'original, en des choses qui en valoient la peine." In a *Lettre* à M. Périer, 1676, which may be seen in the *Œuvres de* Pascal, tom. V. p. 429, Leibnitz has given a slight sketch of the general scheme of this lost treatise: he says that Pascal called his inscribed hexagon the *mystic hexagram*. Desargues says that in his time it was called "the Pascal." A great many demonstrations of the property of the *mystic hexagram* have been given by various mathematicians, both geometrically and analytically. Pascal himself, in his *Essais pour les Coniques*, has merely enunciated the property in a converse form. The following is a list of some of the demonstrations.

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* *Essais pour les Coniques*, *Œuvres de* Blaise Pascal, tom. IV. p. 1.

† *Harmonie Universelle.*

‡ Pascal himself also (*Œuvres*, tom. IV. p. 358), in presenting certain works *Celeberrimæ Matheseos Academiæ Parisiensi*, speaks of this one as "Conicorum opus completum, et conica Apollonii et alia innumera unicâ ferè propositione amplectens; quod quidem nondùm sexdecimum ætatis annum assecutus excogitavi, et deinde in ordinem congessi."

‖ *Lettres à* M. M. Remond de Montmort, *Lettre* 2. *Opera*, tom. v. p. 12.

Carnot: *Géométrie de Position*, p. 452.
Gergonne: *Annales de Mathématiques*, tom. IV. p. 379; 1813, 1814.
Dandelin: *Gergonne, Annales de Mathématiques*, tom. XV. p. 396.
Lubbock: *Annals of Philosophy, October*, 1829.
Davies: *Annals of Philosophy, July*, 1842, p. 37.
Rutherford: *Ibid., March*, 1843, p. 168.
Various Mathematicians: *Lady's and Gentleman's Diary*, 1843.
Kirkman: *Ibid.*, 1849.
Cayley: *One solution* (from Chasles), *Cambridge Mathematical Journal, Old Series*, vol. III. p. 211; *another in* vol. IV. p. 18.
Rutherford: *Davies' Hutton*, vol. II. pp. 213–313.
Fenwick: *Mathematician*, vol. I. p. 132.
Weddle: *Ibid.*, vol. II. p. 15.
Salmon: *Treatise on Conic Sections*, p. 212, 1st edition.
Gaskin: *Geometrical Problems*, p. 244.

7. AD, BC, are the diagonals of a quadrilateral inscribed in a conic section: from A, C, are drawn lines AF, CF, to any point F in the arc BD, and, from B, D, are drawn lines BE, DE, to any point E in the arc AC, meeting AF, CF, respectively in G, H. To prove that the line GH will pass through the intersection of AD, BC.

Davies: *Philosophical Magazine*, 1842, vol. XXI. p. 37.

8. Let P_1, P_2, P_3, Q_1, Q_2, Q_3, be six points lying in a conic section; let the areas of the triangles $P_1Q_2Q_3$, $P_1Q_3Q_1$, $P_1Q_1Q_2$, be denoted by A_1, B_1, C_1, and the areas of the triangles, formed by putting P_2, P_3, successively in the place of P_1, be denoted by A_2, B_2, C_2, A_3, B_3, C_3, respectively: then will

$$\frac{1}{A_1}\left(\frac{1}{B_2C_3}-\frac{1}{B_3C_2}\right)+\frac{1}{A_2}\left(\frac{1}{B_3C_1}-\frac{1}{B_1C_3}\right)+\frac{1}{A_3}\left(\frac{1}{B_1C_2}-\frac{1}{B_2C_1}\right)=0.$$

9. Let $ABCD$ be a quadrilateral: and let a conic section be described about it and tangents be drawn at A, B, C, D, forming another quadrilateral. Let the opposite sides of one quadrilateral intersect in P, Q, and of the other in E, F. To prove that P, E, Q, F, are in the same straight line.

Carnot: *Géométrie de Position*, p. 453.

G. W. H., and A. C.: *Cambridge and Dublin Mathematical Journal*, vol. II. p. 238.

10. If, in the perimeter of any conic section, be taken any six points A, B, C, D, E, F, and the straight lines AB, AF, be produced to meet the straight lines DC, DE, produced, in M, N, respectively: to prove that the three straight lines MN, BE, FC, all pass through a single point.

Carnot: *Géométrie de Position*, p. 452.

11. Two conic sections being circumscribed about any given quadrilateral; if, through the extremities of one and the same side of this quadrilateral, be drawn two arbitrary secants; the chords in these curves which terminate in the points in which they are respectively intersected by these secants, will, if produced indefinitely, cut each other in some point on the direction of the opposite side of the quadrilateral.

Numerous very interesting consequences have been deduced by Plucker from this theorem and an analogous one respecting any circumscribed quadrilateral.

Plucker: *Gergonne, Annales de Mathématiques*, tom. XVII. p. 39.

12. If three sides of a variable quadrilateral inscribed in a conic section are always parallel to three given straight lines, to prove that the fourth side will also be parallel to a given line.

O'Brien: *Plane Coordinate Geometry*, p. 178.

13. Having given any convex quadrilateral; 1st, we may always circumscribe about it an infinite number of ellipses, but only two parabolas; 2nd, the locus of the centres of all these ellipses is an hyperbola, the asymptotes of which are respectively parallel to the axes of the two parabolas; 3rd, the conjugates of the diameters of these ellipses, which are parallel to any fixed line, meet all in a single fixed point of the hyperbolic locus of the centres; 4th, the conjugates of the diameters of these ellipses parallel to one of the two asymptotes of the hyperbolic locus of the centres, are parallel to the other asymptote of this hyperbola; 5th, of all these ellipses, the one which approaches nearest to a circle is that of which the conjugate diameters, parallel to the asymptotes of the hyperbola, are at the same time equal conjugate diameters.

Gergonne: *Annales de Mathématiques*, tom. XVIII. p. 100.

SECTION XXI.

Circumscribed Polygons.

1. If any quadrilateral be circumscribed about a conic section, the point of intersection of the two straight lines which join the points in which the conic section is touched by opposite sides of the quadrilateral, will coincide with the point of intersection of its two diagonals.

Let the equations to three sides of the quadrilateral taken in order be $u = 0, \quad v = 0, \quad w = 0.$

Then the equation to the conic section will be

$$(lu)^{\frac{1}{2}} + (mv)^{\frac{1}{2}} + (nw)^{\frac{1}{2}} = 0.^*$$

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* This equation to a conic section inscribed in a triangle is substantially the same as that given by Bobillier in Gergonne's *Annales de Mathématiques*, tom. XVIII. p. 325.

The equation to any other tangent of the conic section, which we will take for the fourth side of the quadrilateral, will be

$$\lambda u + \mu v + \nu w = 0,$$

λ, μ, ν, being connected with l, m, n, by the equation

$$\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0 \text{........................} (1).$$

The equations to the points in which the lines $u = 0$, $w = 0$, touch the conic section, are, respectively,

$$\left\{\begin{matrix} u = 0 \\ mv = nw \end{matrix}\right\} \text{ and } \left\{\begin{matrix} w = 0 \\ mv = lu \end{matrix}\right\}.$$

Hence the equation to the line, which joins the points of contact of two opposite sides of the quadrilateral, is

$$mv = lu + nw \text{........................} (2).$$

Again, the equations to the two diagonals are

$$\mu v + \nu w = 0 \text{........................} (3),$$

and

$$\lambda u + \mu v = 0 \text{........................} (4).$$

Eliminating u, v, w, between the equations (2), (3), (4), we obtain the equation (1): the two diagonals therefore intersect in the line which joins two opposite points of contact. The quadrilateral being of a perfectly general form, it follows that the intersection of the two diagonals must take place also in the line joining the other two opposite points of contact. The four lines therefore intersect in a single point.

Carnot: *Géométrie de Position*, p. 454.

Peschier, Rochat, Ferriot, Fornier: *Gergonne*, *Annales de Mathématiques*, tom. III. p. 161.

Hearn: *Researches on Curves of the Second Order*, p. 52.

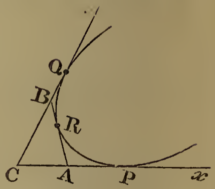
2. The sides of a triangle ABC are tangents to a given conic section, CA touching it in P, CB in Q, and AB in R. CA, CB, are fixed in position, AB being variable. To prove that the ratio

$$\frac{AP}{AR} : \frac{BQ}{BR}$$

is a constant quantity.

Let CP, CQ, produced indefinitely, be taken as axes of coordinates, and let the equation to ARB in any position be

$$w = ax + by + c = 0.$$



Then, l and m being arbitrary constants, the equation to the conic section will be

$$l^2x^2 + m^2y^2 + w^2 = 2lxw + 2myw + 2lmxy.$$

Putting $y = 0$ in the equation to the conic section, we see that, at the point P,

$$lx = ax + c, \quad x = \frac{c}{l - a}.$$

But, at A, putting $y = 0$ in the equation to AB, we have

$$x = -\frac{c}{a}.$$

Hence $$AP = \frac{cl}{a(l - a)}.$$

Again, at R, we have $$w = 0,$$
$$lx = my,$$

and therefore $$x = -\frac{mc}{ma + bl}, \quad y = -\frac{lc}{ma + bl}.$$

Hence, for the length AR, we have, ω denoting the angle xCy,

$$(AR)^2 . \frac{(ma + bl)^2}{(l - a)^2} = (AP)^2 . (a^2 + b^2 - 2ab \cos \omega).$$

Similarly

$$(BR)^2 . \frac{(ma + bl)^2}{(m - b)^2} = (BQ)^2 . (a^2 + b^2 - 2ab \cos \omega).$$

Hence $$(l - a)^2 . \left(\frac{AP}{AR}\right)^2 = (m - b)^2 . \left(\frac{BQ}{BR}\right)^2.$$

But, the conic section being given, the ratio of the coefficients of x and y in its equation must be invariable for all positions of R: hence

$$\frac{l - a}{m - b} = \text{a constant quantity.}$$

Consequently $$\frac{AP}{AR} : \frac{BQ}{BR} = \text{a constant quantity.}$$

Cor. This proposition is an obvious deduction from the theorem that the products of the alternate segments of the sides of a polygon circumscribing a conic section are equal.

3. If any hexagon be circumscribed about a conic section, the three diagonals which join its opposite angles will all pass through a single point.

Let the equations to three alternate sides of the hexagon be

$$u = 0 \ldots\ldots\ldots (1), \quad v = 0 \ldots\ldots\ldots (3), \quad w = 0 \ldots\ldots\ldots (5).$$

Then the equation to the conic section will be

$$(lu)^{\frac{1}{2}} + (mv)^{\frac{1}{2}} + (nw)^{\frac{1}{2}} = 0,$$

l, m, n, being arbitrary constants.

The equations to the three other alternate sides will be

$$\lambda u + \mu v + \nu w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\lambda' u + \mu' v + \nu' w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$\lambda'' u + \mu'' v + \nu'' w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where the coefficients of u, v, w, are subject to the relations

$$\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{l}{\lambda'} + \frac{m}{\mu'} + \frac{n}{\nu'} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$\frac{l}{\lambda''} + \frac{m}{\mu''} + \frac{n}{\nu''} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

The equations to the diagonals

$$\{(1, 2), (4, 5)\}, \quad \{(2, 3), (5, 6)\}, \quad \{(3, 4), (6, 1)\},$$

will, as is evident by inspection, be respectively

$$\frac{u}{\mu'\nu} + \frac{v}{\nu\lambda'} + \frac{w}{\lambda'\mu} = 0,$$

$$\frac{w}{\lambda\mu''} + \frac{u}{\mu''\nu} + \frac{v}{\nu\lambda''} = 0,$$

$$\frac{v}{\nu''\lambda'} + \frac{w}{\lambda'\mu''} + \frac{u}{\mu''\nu'} = 0.$$

If between these three equations we eliminate u, v, w, by cross multiplication, we shall obtain an equation coincident with

$$\frac{1}{\lambda}\left(\frac{1}{\mu'\nu''}-\frac{1}{\nu'\mu''}\right)+\frac{1}{\lambda'}\left(\frac{1}{\mu''\nu}-\frac{1}{\nu''\mu}\right)+\frac{1}{\lambda''}\left(\frac{1}{\mu\nu'}-\frac{1}{\mu'\nu}\right)=0:$$

this equation however results from the elimination of l, m, n, between the equations (7), (8), (9). Hence the three diagonals must all pass through a single point.

"M. Brianchon, élève, m' a remis l'analyse d'un mémoire dans lequel il prouve par la géométrie seule, plusieurs propriétés des courbes et des surfaces du second degré, dont quelques-uns lui appartiennent. Ce mémoire est destiné pour le Journal de l'Ecole: j'en extrais la proposition suivante: 'si tous les côtès d'un hexagone quelconque touchent une même courbe du second degré, les trois diagonales, prolongées s'il le faut, se croisent en un même point.'" Hachette: *Correspondance sur l'Ecole Polytechnique*, tom. I. p. 151.

Brianchon: *Journal de l'Ecole Polytechnique*, tom. IV.

" *Correspondance sur l'Ecole Polytechnique*, tom. I. p. 307; tom. II. p. 383.

Carnot: *Essai sur la Théorie des Transversales.*

Gergonne: *Annales de Mathématiques*, tom. IV. p. 383.

Dandelin: *Gergonne, Annales de Mathématiques*, tom. XV. p. 396.

Davies: *Annals of Philosophy*, *November*, 1826.

Lubbock: *Ibid.*, *October*, 1829.

Fenwick: *Ibid.*, *March*, 1843, p. 167.

Frost: *Cambridge Mathematical Journal*, vol. IV. p. 277.

The student is referred also to a memoir by Coste in the *Annales de Mathématiques*, tom. X. p. 261, where he will find numerous properties of the parabola deduced from Pascal's and Brianchon's general theorems concerning the inscribed and circumscribed hexagons of a conic section.

4. If ABC be a triangle circumscribed about a conic section, α, β, γ, being the points in which the sides BC, CA, AB, touch it, to prove that $A\alpha$, $B\beta$, $C\gamma$, all pass through a single point.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 323.

Salmon: *Treatise on Conic Sections*, p. 236.

5. Two triangles being the one inscribed and the other circumscribed in relation to a conic section, so that the summits of the inscribed triangle are the points of contact of the circumscribed; the points of concourse of the sides of the inscribed triangle with the respectively opposite sides of the circumscribed, lie all three in a single straight line.

Bobillier: *Gergonne, Annales de Mathématiques*, tom. XVIII. p. 323.
Salmon: *Treatise on Conic Sections*, p. 235.

6. Two conic sections being inscribed in the same quadrilateral; if, in the two sides of one of the summits of this quadrilateral, be taken arbitrarily two points through each of which we draw tangents to the two curves; the point of concourse of the pair of tangents to one of the curves, that of the pair of tangents to the other curve, and the opposite summit of the quadrilateral, will all three lie in a straight line.

Plucker: *Gergonne, Annales de Mathématiques*, tom. XVII. p. 39.

7. If a conic section be circumscribed by a triangle ABC, and lines $AA'\alpha$, $BB'\beta$, $CC'\gamma$, be drawn from the summits A, B, C, to the points of contact α, β, γ, of the opposite sides, meeting the conic section in three points A', B', C': to prove that tangents drawn to the curve at the points A', B', C', will intersect each other, two and two, in the lines $A\alpha$, $B\beta$, $C\gamma$. Also, to prove that the intersections of the tangents at A', B', C', respectively, with the sides BC, CA, AB, of the circumscribed triangle, will all lie in a single straight line.

Leybourn: *Mathematical Repository*, *New Series*, vol. II. p. 153.

8. To find the equation to a conic section inscribed in a triangle so as to touch the three sides at their middle points.

Taking $(x_2 - x_1)y - (y_2 - y_1)x - x_2y_1 + x_1y_2 = 0$, as the general type of the equation to a straight line, then, the equations to the sides being

$$u = 0, \quad v = 0, \quad w = 0,$$

we shall have for the equation to the required conic section

$$u^{\frac{1}{2}} + v^{\frac{1}{2}} + w^{\frac{1}{2}} = 0.$$

SECTION XXII.

Problems relating to several Curves.

1. If there be, in one plane, any two lines of the second order, such that the principal diameters of the one coincide with the tangent and normal at any given point of the other; the line which joins the points, in which any two conjugate diameters of the one intersect the other, will pass through an invariable point of the given normal.

Taking the tangent and normal at the given point as axes of x and y, we shall have for the equation to the one conic section

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, for the equation to the other,

$$ax^2+by^2+2cxy+2fy=0\ldots\ldots\ldots\ldots\ldots(2).$$

The equations to any two conjugate diameters of (1) will be

$$y=mx,\quad y=-\frac{\beta^2}{m\alpha^2}x,$$

and the equation to both will be

$$(y-mx)\left(y+\frac{\beta^2}{m\alpha^2}x\right)=0,$$

whence

$$x^2=\frac{\alpha^2}{\beta^2}y^2+\left(\frac{1}{m}-\frac{m\alpha^2}{\beta^2}\right)xy\ldots\ldots\ldots\ldots\ldots(3).$$

Substituting in (2) the expression for x^2, given by (3), we have, dividing by y,

$$\left(\frac{a}{m}-\frac{ma\alpha^2}{\beta^2}+2c\right)x+\left(\frac{a\alpha^2}{\beta^2}+b\right)y+2f=0\ldots\ldots(4),$$

which will be the equation to the line joining the points in which the conic section (2) is cut by the two conjugate diameters of (1).

Putting $x=0$ in (4), we see that

$$y=-\frac{2f\beta^2}{a\alpha^2+b\beta^2},$$

which determines a constant point in the normal through which all such lines pass.

Cor. By the theory of poles and polars, it hence follows that the pairs of tangents, at the extremities of all lines the equations of which are of the form (4), all intersect in a single straight line.

Frégier: *Gergonne, Annales de Mathématiques*, tom. VI. p. 321.

2. If two conic sections (A) and (B) are so placed that the centre of (B) lies in the perimeter of (A); all the chords of (A), which terminate at its intersections with the prolongations of two conjugate diameters of (B), will intersect each other in a single point, situated in the conjugate to that diameter of (B) which is a tangent to (A).

Let the tangent to (A), at the centre of (B), be taken as the axis of x, and the diameter of (B), which is conjugate to this tangent, as that of y.

Then the equation to (A) will be of the form

$$ax^2 + by^2 + 2cxy + fy = 0 \text{.................} (1).$$

Also the equations to any two conjugate diameters of (B) will be

$$y = mx, \quad y = m'x,$$

where mm' is equal to some constant quantity k.

The equation to both these conjugate diameters will be

$$\begin{aligned}(y - mx)(y - m'x) &= y^2 - (m + m')\,xy + mm'x^2 \\ &= y^2 - (m + m')\,xy + kx^2 = 0 \text{...........} (2).\end{aligned}$$

At the intersections of (1) and (2), we have, eliminating x^2,

$$(kb - a)\,y + \{2kc + (m + m')\,a\}\,x + kf = 0,$$

which is the equation to the chord of (A) joining its intersections with the two conjugate diameters of B.

Putting $x = 0$, we see that

$$y = \frac{kf}{a - kb},$$

a result which, being free from m and m', establishes the proposition.

Frégier: *Gergonne, Annales de Mathématiques*, tom. VII. p. 95.

3. The equations to two conic sections being

$$Ax^2 + 2Cxy + By^2 + 2A'x = 0,$$
$$ax^2 + 2cxy + by^2 + 2a'x = 0;$$

to find the condition that the lines joining the origin with their points of intersection may be perpendicular to each other.

Multiplying the former equation by a', and the latter by A', and subtracting, we get

$$(Aa' - A'a)\,x^2 + 2\,(Ca' - A'c)\,xy + (Ba' - A'b)\,y^2 = 0,$$

the equation to the two joining lines.

The product of the tangents of their inclinations to the axis of x will be equal to

$$\frac{Aa' - A'a}{Ba' - A'b}:$$

but, if the two lines be at right angles to each other, their product must be equal to -1. Hence, for the required condition, we have $Aa' - A'a = -Ba' + A'b$,

or
$$\frac{A+B}{a+b} = \frac{A'}{a'}.$$

4. If in the plane of a conic section U, we describe any two circles A, A', and two conic sections B, B', of which the former passes through the four points of intersection (real or imaginary) of the former circle and the conic section U, and the latter through the four points of intersection of the latter circle and of the conic section U; to prove that

(1), the two conic sections B, B', cut each other in four points which lie in a circle; and, (2), that this circle passes through the points of intersection of the two proposed circles.

Let U's equation be $u = 0$,
A's .... $\alpha = 0$,
A''s...............$\alpha' = 0$.

Then, λ and λ' being any constant multipliers,
B's equation will be $\lambda u + \alpha = 0$,
B''s.....................$\lambda' u + \alpha' = 0$.

At the intersections of B and B' we have

$$\lambda'\alpha - \lambda\alpha' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

which is of the form of the equation to a circle: the four intersections of B, B', therefore lie in a circle.

Again, since the equation (1) is satisfied by all values of x and y which satisfy A's and A''s equations simultaneously, we see that the circle (1) passes through all the intersections of A and A'.

5. To prove that two common chords, real or imaginary, of two confocal conic sections, pass through the intersection of the directors.

Any two such conic sections may be represented by the equations

$$\frac{c}{r} = 1 + e \cos(\theta + \alpha) \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{c'}{r} = 1 + e' \cos(\theta + \alpha') \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now, the coordinates of any point in (1) being (r, θ), those of a coincident point in (2) must be

$$\{r,\ 2n\pi + \theta\} \quad \text{or} \quad \{-r,\ (2n+1)\pi + \theta\},$$

n being any integer.

Hence, where (1) and (2) intersect, we have, either

$$\frac{c - c'}{r} = e\cos(\theta + \alpha) - e'\cos(2n\pi + \theta + \alpha'),$$

or
$$\frac{c + c'}{r} = e\cos(\theta + \alpha) - e'\cos(2n\pi + \pi + \theta + \alpha').$$

Thus every point of intersection lies in one of the two chords

$$\frac{c \pm c'}{r} = e\cos(\theta + \alpha) \pm e'\cos(\theta + \alpha'),$$

both of which pass through the intersection of the directors, the equations to which are

$$\frac{c}{r} = e\cos(\theta + \alpha), \quad \frac{c'}{r} = e'\cos(\theta + \alpha').$$

This proposition may be proved also in the following manner. Let the equation to the director of one of the conic sections, the focus being the origin, be

$$lx + my = \delta,$$

l, m, being its direction-cosines.

Then, e denoting the eccentricity, the equation to the curve will be

$$x^2 + y^2 = e^2 (lx + my - \delta)^2.$$

In like manner, the equation to the other conic section will be of the form

$$x^2 + y^2 = e'^2 (l'x + m'y - \delta')^2.$$

At the intersection of the two curves

$$e (lx + my - \delta) = \pm e' (l'x + m'y - \delta'),$$

which is therefore the equation to two chords of intersection. The form of the equation shews that both chords pass through the intersection of the directors.

6. A conic section is cut in four points by a circle, and two straight lines, each passing through two of the points of intersection, are taken as axes of coordinates: to prove that the equation to the conic section will be of the form

$$x^2 + 2cxy + y^2 + 2a'x + 2b'y + c' = 0.$$

7. If any number of circles be described so as to touch a given curve of the second order in one given point: to prove that they will each of them generally cut it in two others; and that, if a chord be drawn through the two points of section for each circle, such chords will be all parallel to one another.

8. If two conic sections meet each other in more than four points, to prove that they will coincide.

De la Hire: *Sectiones Conicæ*, lib. II. prop. 32.

9. If a parabola, represented by the equation $y^2 = lx$, is intersected in four points by any conic section: to prove that,

y_1, y_2, y_3, y_4, being the ordinates of the intersections,

$$y_1 + y_2 + y_3 + y_4 = 0.$$

De la Hire: *Sectiones Conicæ*, lib. v. prop. 31.

10. To prove that two conic sections of the same species, the axes of which are proportional in magnitude and parallel in direction, cannot have more than two points in common.

Lamé: *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*, p. 39.

SECTION XXIII.

Intersections of Conic Sections. Common Chords.

If $S=0$ be the equation to a conic section, that to any other conic section cutting it at the ends, real or imaginary, of two chords $u=0$, $v=0$, will be

$$S = kuv,$$

k being an arbitrary constant.

Salmon: *Treatise on Conic Sections*, p. 199.

1. To find the diameter of a circle described about a semi-ellipse bounded by its axis minor.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equations to the common chord and common tangent, the latter of which is merely a degenerate case of a chord, are

$$x = 0, \quad x = a:$$

hence the equation to the circle is included in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = kx(x-a).$$

Since this is the equation to a circle, the coefficients of x^2 and y^2 must be the same: hence

$$k = \frac{1}{a^2} - \frac{1}{b^2},$$

and therefore the equation becomes

$$x^2 + y^2 = b^2 + \frac{a^2 - b^2}{a} x,$$

the diameter of the circle being therefore equal to $\dfrac{a^2 - b^2}{a}$.

2. If any number of conic sections have four points in common: in whatever direction parallel diameters be drawn in them, the conjugate diameters will all meet in a single point.

Any one of the system of conic sections may be represented by the equation

$$S = kuv,$$

where k is an arbitrary constant, varying as we pass from one of the curves to another; where

$$S = ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c',$$

$$u = \alpha x + \beta y + \gamma,$$

and

$$v = \alpha' x + \beta' y + \gamma';$$

$u = 0$, and $v = 0$, denoting the equations to two common chords of all the curves.

Now, if the equation to a conic section be written in the form

$$Ax^2 + By^2 + 2Cxy + 2A'x + 2B'y + C' = 0,$$

the equation to a diameter, conjugate to one which is parallel to a chord $y = mx$, will be

$$Ax + Cy + A' + m(By + Cx + B') = 0:$$

hence, in the present case, if we put for A, B, C, A', B', their values, the conjugate diameter will have an equation of the form

$$U + kV = 0,$$

where U, V, are linear functions of x and y, not involving k.

This diameter evidently passes through a point defined by the equations

$$U = 0, \quad V = 0.$$

Since these equations are independent of k, this point is invariable in position for every such conjugate diameter.

Lamé: *Mémoires de l'Académie des Sciences de Paris, Dec.* 1816.
" *Gergonne, Annales de Mathématiques*, tom. VII. p. 233.

3. If a circle cut a conic section, to prove that the chords joining the points of intersection are equally inclined to the axis of the conic section.

Leslie: *Geometrical Analysis*, p. 302.

4. If two conic sections, the axes of which are parallel, intersect one another in four points, to prove that the sides AB, CD, and also the sides BC, DA, of a quadrilateral $ABCD$, formed by any four common chords AB, BC, CD, DA, are equally inclined to either axis.

Elliot: *Lady's and Gentleman's Diary*, 1851.

SECTION XXIV.

Double Contact.

If two conic sections touch each other at two points, they are said to have double contact with each other.

Let $u = 0$ be the equation to a conic section, and let another conic section touch it at the extremities, real or imaginary, of a chord $v = 0$: then the equation to the other conic section will be

$$u = kv^2,$$

k being an arbitrary constant.

Salmon: *Treatise on Conic Sections*, p. 200.

1. To find the equation to an ellipse having double contact with an hyperbola represented by the equation

$$xy = c^2,$$

the asymptotes of the hyperbola coinciding with conjugate diameters of the ellipse.

Let the equation to the chord of contact be

$$\frac{x}{m} + \frac{y}{m} = 1:$$

then the equation to the ellipse will be

$$c^2 - xy = k^2 \left(\frac{x}{m} + \frac{y}{n} - 1\right)^2 \ldots\ldots\ldots\ldots\ldots (1).$$

Since this ellipse is referred to conjugate diameters, we must have

$$\left.\begin{array}{r} \dfrac{k^2}{m} = 0 \\ \dfrac{k^2}{n} = 0 \\ 2\dfrac{k}{m} \cdot \dfrac{k}{n} = -1 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These three equations shew that $k = 0$, $\frac{k}{m}$ and $\frac{k}{n}$ being finite. The equation (1) therefore becomes

$$\frac{k^2}{m^2} x^2 + \frac{k^2}{n^2} y^2 = c^2,$$

or, a, b, being the conjugate semi-diameters,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

a and b being, by virtue of the third of the equations (2), connected together by the relation

$$a^2 b^2 = 4c^4.$$

2. To find the equation to a circle inscribed in a semi-ellipse bounded by its minor axis.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation to the chord of contact, as is evident from symmetry, will be of the form

$$x = c:$$

hence the equation to the circle will be included in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=k(x-c)^2.$$

Since the circle touches the axis of y, we must have

$$kc^2=-1;$$

hence
$$c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=2cx-x^2,$$

$$\left(\frac{c^2}{a^2}+1\right)x^2+\frac{c^2}{b^2}y^2=2cx.$$

The curve being a circle, the coefficients of x^2 and y^2 must be equal: hence
$$c^2=\frac{a^2b^2}{a^2-b^2},$$

and therefore the equation to the required circle is

$$x^2+y^2=\frac{2b}{a}(a^2-b^2)^{\frac{1}{2}}.x.$$

SECTION XXV.

Conical Loci.

1. To determine the locus of a point, the algebraical sum of the distances of which from two given straight lines and from a given point, shall be constant.

Let the equations to the two given lines be

$$x\cos\varepsilon+y\sin\varepsilon-\delta=0,$$
$$x\cos\varepsilon'+y\sin\varepsilon'-\delta'=0;$$

and (a, b) the coordinates of the given point.

Then, by the condition of the problem, the equation to the required locus will be, k denoting a constant,

$$x(\cos\varepsilon+\cos\varepsilon')+y(\sin\varepsilon+\sin\varepsilon')-\delta-\delta'+\{(x-a)^2+(y-b)^2\}^{\frac{1}{2}}=k;$$

or, writing c for $k+\delta+\delta'$, α for $\frac{\varepsilon-\varepsilon'}{2}$, and β for $\frac{\varepsilon+\varepsilon'}{2}$,

$$2x\cos\alpha\cos\beta+2y\cos\alpha\sin\beta+\{(x-a)^2+(y-b)^2\}^{\frac{1}{2}}=c,$$

and therefore, clearing the equation of the radical sign,

$$\begin{aligned}(x-a)^2+(y-b)^2 = {} & 4\cos^2\alpha\cos^2\beta . x^2 + 8xy\cos^2\alpha\sin\beta\cos\beta \\ & + 4y^2\cos^2\alpha\sin^2\beta \\ & - 4c\,(x\cos\alpha\cos\beta + y\cos\alpha\sin\beta) + c^2,\end{aligned}$$

$$\begin{aligned}& x^2\,(4\cos^2\alpha\cos^2\beta - 1) + 8xy\cos^2\alpha\sin\beta\cos\beta + y^2\,(4\cos^2\alpha\sin^2\beta - 1) \\ & + 2x\,(a - 2c\cos\alpha\cos\beta) + 2y\,(b - 2c\cos\alpha\sin\beta) = a^2 + b^2 - c^2.\end{aligned}$$

The curve will therefore be an ellipse, a parabola, or an hyperbola, respectively, accordingly as

$$64\cos^4\alpha\sin^2\beta\cos^2\beta \lesseqqgtr 4\,(4\cos^2\alpha\cos^2\beta - 1)\,(4\cos^2\alpha\sin^2\beta - 1),$$

$$0 \lesseqqgtr 1 - 4\cos^2\alpha,$$

$$\cos^2\alpha \lesseqqgtr \tfrac{1}{4}.$$

Lhuilier: *Gergonne*, *Annales de Mathématiques*, tom. II. p. 173.

2. To find the locus of the pole of a given circle, the corresponding polar of which is always a tangent to another given circle.

Let the radius of the circle, to which the polar is a tangent, be r: then, the centre of this circle being the origin, the equation to the polar will be $x\cos\theta + y\sin\theta = r$....................(1).

Let the corresponding pole be at the point h, k; then, a, b, being the coordinates of the centre and c the radius of the other circle, the equation to the polar will be

$$x\,(h-a) + y\,(k-b) = c^2 - a^2 - b^2 + ha + kb \ldots\ldots(2).$$

Since (1) and (2) must be coincident, we have

$$\frac{r}{\cos\theta}\,(h-a) = c^2 - a^2 - b^2 + ha + kb = \frac{r}{\sin\theta}\,(k-b),$$

and therefore

$$(c^2 - a^2 - b^2 + ha + kb)^2 = r^2\,\{(h-a)^2 + (k-b)^2\},$$

which is the equation to the locus of (h, k). This locus is an ellipse, parabola, or hyperbola, accordingly as r is greater than, equal to, or less than $(a^2+b^2)^{\frac{1}{2}}$, the distance between the centres of the two circles.

L'Hospital: *Traité Analytique des Sections Coniques*, p. 273.

Puissant: *Recueil de diverses propositions de Géométrie*, p. 219, *troisième édition*.

3. If a straight line DCP be made to revolve about C, and intersect a curve in as many points P, P_1, P_2,...... as it has dimensions, and if $\frac{1}{CD^2}$ be made always equal to

$$\frac{1}{CP^2}+\frac{1}{CP_1^2}+\frac{1}{CP_2^2}+\ldots\ldots:$$

to find the locus of the point D.

Let the equation to the intersected curve be

$$1+ax+by+a'x^2+b'y^2+c'xy+\ldots\ldots=0.$$

Put $x=r\cos\theta$, $y=r\sin\theta$: then

$$1+r(a\cos\theta+b\sin\theta)+r^2(a'\cos^2\theta+b'\sin^2\theta+c'\sin\theta\cos\theta)+\ldots=0.$$

Hence, by the theory of equations,

$$\frac{1}{CP^2}+\frac{1}{CP_1^2}+\frac{1}{CP_2^2}+\ldots\ldots=(a\cos\theta+b\sin\theta)^2 \\ -2(a'\cos^2\theta+b'\sin^2\theta+c'\sin\theta\cos\theta).$$

Consequently, putting $CD=\rho$, we obtain

$$\frac{1}{\rho^2}=(a^2-2a')\cos^2\theta+2(ab-c')\sin\theta\cos\theta+(b^2-2b')\sin^2\theta,$$

which is the polar equation to a conic section of which C is the centre.

4. To find the locus of the summit of an angle of given magnitude, circumscribed on a parabola.

Let ε be the given angle; then, the equations to its two sides being

$$\frac{2m}{r}=\cos\theta+\cos(\theta-\alpha)\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{2m}{r}=\cos\theta+\cos(\theta-\alpha')\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

we shall have also

$$\alpha'-\alpha=2\varepsilon\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

At the intersection of (1) and (2), there is

$$\cos(\theta-\alpha)=\cos(\theta-\alpha'),$$

and therefore

$$\theta=\tfrac{1}{2}(\alpha+\alpha')\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, adding (1) and (2), we get

$$\frac{4m}{r} = 2\cos\theta + 2\cos\left(\theta - \frac{\alpha + \alpha'}{2}\right)\cos\frac{\alpha' - \alpha}{2}:$$

hence, from (3) and (4), we get

$$\frac{2m}{r} = \cos\theta + \cos\varepsilon,$$

or

$$r = \frac{2m}{\cos\varepsilon + \cos\theta},$$

the equation to the locus of the summit of the angle, which is therefore a conic section confocal with the parabola.

L'Hospital: *Traité Analytique des Sections Coniques*, liv. VIII.

5. The distance of a point P from the circumference of a circle is to its distance from a fixed diameter as n to 1: to find the locus of P.

The fixed diameter being taken as the axis of y, and a perpendicular diameter as that of x, the locus will be defined by the equation

$$x^2 + y^2 = (nx + a)^2,$$

where a denotes the radius of the circle.

6. From a point M in a given straight line AB, a perpendicular MP is erected, such that

$$MP^2 \propto AM.MB;$$

to find the equation to the locus of P.

Let $AB = 2a$, $MP = y$, and, C being the middle point of AB, let $CM = x$. The locus will be a conic section defined by the equation

$$y^2 = \pm\lambda(x^2 - a^2),$$

λ being a constant quantity.

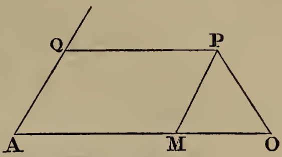
Pappus: *Mathematicæ Collectiones à Commandino*, lib. VII. p. 298.

7. QAO is a given angle, O a given point: P is a point such that, PO being joined, and PQ drawn parallel to OA,

$$PO : PQ :: m : n.$$

To find the locus of P.

Let PM be drawn parallel to QA, cutting OA in M. Let $OA = a$, $OM = x$, $PM = y$, $\angle QAO = \omega$. Then the locus of P will be a conic section defined by the equation



$$x^2(n^2 - m^2) + n^2y^2 - 2n^2xy\cos\omega$$
$$+ 2m^2ax = m^2a^2.$$

Newton: *Arithmetica Universalis*, prob. 24.

8. To determine the locus of the poles of a given straight line with respect to a series of circles touching two given straight lines.

Let the two given tangents be taken as axes of coordinates, and let the equation to the given polar be

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1.$$

Then, ω denoting the angle between the axes of coordinates, the equation to the required locus will be

$$\{\alpha x - \beta y + (\alpha y - \beta x)\cos\omega\}^2$$
$$= (\alpha - \beta).[(x + y)\{\alpha x - \beta y + (\alpha y - \beta x)\cos\omega\} + \alpha\beta(x - y)(1 - \cos\omega)].$$

Poncelet: *Gergonne, Annales de Mathématiques*, tom. XII. p. 234

9. Three straight lines revolve about three given points not in the same straight line, and intersect one another in three points: if the loci of two of these intersections be given straight lines, to find the locus of the third.

Let O, A, B, be the three points. Let OA, OB, produced indefinitely, be taken as axes of x, y, respectively. Let $OA = a$, $OB = b$.

Let the lines through A, O, intersect in the given line

$$y = \alpha x + \beta,$$

and those through B, O, in the given line

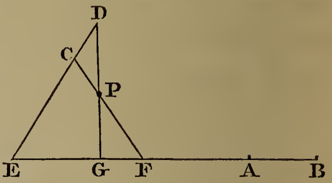
$$y = \alpha' x + \beta':$$

then the locus of the intersection of the lines through A, B, will be a conic section defined by the equation

$$(b - \beta').(a\alpha + \beta)\,xy = \{ay + \beta(x - a)\}.\{\alpha' bx - \beta'(y - b)\}.$$

10. Through two given points C, D, is drawn a straight line DCE, cutting a given straight line BA, produced indefinitely, in E. Two points F, G, are taken in EA such that

$$AF.BG = c^2,$$



c being a constant length. CF, DG, are joined. To find the locus of P, their point of intersection.

Taking EAB, ECD, produced indefinitely as axes of x, y, respectively, and putting $CE = k$, $DE = k'$, $EA = a$, $EB = a'$, we shall have for the equation to the required locus

$$c^2 (y - k)(y - k') = \{k(x - a) + ay\}.\{k'(x - a') + a'y\}.$$

Leybourn: *Mathematical Repository*, *New Series*, vol. I. p. 106.

11. The sides BC, CA, AB, of a triangle pass always through three given points (h, k), (h', k'), (h'', k''); and the angular points A, B, move along two given right lines Ox, Oy, respectively: to find the locus of C.

The lines Ox, Oy, being taken as axes of coordinates, the required locus will be a conic section defined by the equation

$$\frac{h''(k' - y)}{k'x - h'y} + \frac{k''(h - x)}{hy - kx} = 1.$$

O'Brien: *Plane Coordinate Geometry*, p. 176.

12. ABC is a given triangle: to find the locus of a point P, such that, MP, a perpendicular to the side BC, being produced to cut the sides of the triangle BA, CA, produced if necessary, in the points q, r, respectively,

$$(MP)^2 \propto Pq.Pr.$$

Draw AO at right angles to the line BC, cutting it in O. Take OB, OA, produced indefinitely, as axes of x, y, respectively. Let $AO = h$, $BO = b$, and $CO = c$. Then the required locus will be a conic section defined by the equation

$$\frac{h^2}{bc}x^2 + (\lambda - 1)y^2 + h\left(\frac{1}{c} - \frac{1}{b}\right)xy - h^2\left(\frac{1}{c} - \frac{1}{b}\right)x + 2hy - h^2 = 0,$$

where λ denotes a constant quantity.

Carnot: *Géométrie de Position*, p. 443.

This problem is merely a particular case of one known by the name of Pappus'* Problem, solved by Des Cartes.† The more general problem is stated by Pappus, in the form of a theorem, in the following words, extracted from Commandine's translation: "Si positione datis tribus rectis lineis ab uno, et eodem puncto ad tres lineas in datis angulis rectæ lineæ ducantur, et data sit proportio rectanguli contenti duabus ductis ad quadratum reliquæ: punctum contingit positione datum solidum locum, hoc est, unam ex tribus conicis sectionibus. Et si ad quatuor rectas lineas positione datas in datis angulis lineæ ducantur; et rectanguli duabus ductis contenti ad contentum duabus reliquis proportio data sit: similiter punctum datam coni sectionem positione continget. Si quidem igitur ad duas tantum, locus planus ostensus est. Quòd si ad plures, quam quatuor, punctum continget locos non adhuc cognitos, sed lineas tantum dictas; quales autem sint, vel quam habeant proprietatem, non constat. Earum unam, neque primam, et quæ manifestissima videtur, composuerunt ostendentes utilem esse, propositiones autem ipsarum hæ sunt.

Si ab aliquo puncto ad positione datas rectas lineas quinque ducantur rectæ lineæ in datis angulis, et data sit proportio solidi parallelepipedi rectanguli, quod tribus ductis lineis continetur ad solidum parallelepipedum rectangulum, quod continetur reliquis duabus, et data quapiam linea, punctum positione datam lineam continget. Si autem ad sex, et sit data proportio solidi tribus lineis contenti ad solidum, quod tribus reliquis continetur; rursus punctum continget positione datam lineam. Quod si ad plures quam sex, non adhuc habent dicere, data sit proportio cujuspiam contenti quatuor lineis ad id quod reliquis continetur, quoniam non est aliquid contentum pluribus quam tribus dimensionibus."

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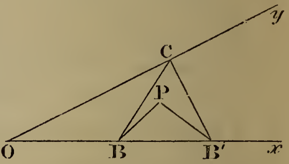
* Pappus: *Mathematicæ Collectiones à Commandino*, lib. VII. p. 165.

† Descartés: *Géométrie*, livre II. p. 30, &c.

13. B, B', are two fixed points in a given straight line Ox; C is a moveable point in the straight line Oy. To find the locus of a point P, such that, the straight lines BC, BP, $B'C$, $B'P$, being drawn, the angles CBP, $CB'P$, are always of given magnitudes.



Take Ox, Oy, as axes of x and y, and let $OB = b$, $OB' = b'$, $\angle xOy = \omega$, $\angle CBP = \alpha$, $\angle CB'P = \alpha'$. Then the required locus will be a conic section defined by the equation

$$b \left\{\cot\omega + \frac{(b' - x)\cos\alpha' - y\cos(\omega - \alpha')}{(b' - x)\sin\alpha' + y\sin(\omega - \alpha')}\right\}$$
$$= b' \left\{\cot\omega + \frac{(b - x)\cos\alpha - y\cos(\omega + \alpha)}{-(b - x)\sin\alpha + y\sin(\omega + \alpha)}\right\}.$$

L'Hospital: *Traité Analytique des Sections Coniques*, p. 281.
O'Brien: *Plane Coordinate Geometry*, p. 179.

14. Having given one side of a triangle and the product of the tangents of the adjacent angles, to find the locus of the vertex.

Let $\pm m$ denote the given product and $2a$ the given side: then, this side being taken as the axis of x, and its middle point as the origin of rectangular coordinates, the required locus will be a conic section represented by the equation

$$y^2 = \pm m(a^2 - x^2).$$

Lardner: *Algebraic Geometry*, p. 116.

15. From a point P, in one given straight line, is drawn PM at right angles to another given straight line AMx: from A, which is a given point, is drawn AQ to a point Q, in PM, such that $AQ = MP$: to find the locus of Q.

If the equation to the former given line, referred to Ax, and Ay, at right angles to Ax, as axes, be

$$y = \alpha x + \beta,$$

then the polar equation to Q, A being the pole and Ax the prime radius vector, will be

$$r = \frac{\beta}{1 - \alpha \cos\theta}.$$

Lardner: *Algebraic Geometry*, p. 150.

16. A straight line of given length moves parallel to itself in a given direction, and two other straight lines revolve round given points and pass always through the extremities of the first line: to find the locus of the intersection of the revolving lines.

Let the axis of x be taken parallel to the moving line, the axis of y always bisecting this line. Let (a, b), (a', b'), be the two given points; and suppose the length of the moving line to be $2c$. Then the equation to the required locus will be

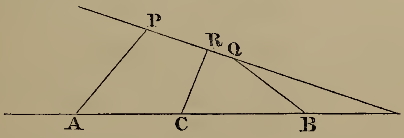
$$b' - b = (m - a) \cdot \frac{y - b}{x - a} + (m + a') \cdot \frac{y - b'}{x - a'}.$$

SECTION XXVI.

Envelopes.

1. A, B, are the centres of two equal circles, and AP, BQ, are two radii which are always perpendicular to each other: to find the curve which is always touched by the right line PQ.

Let C, the middle point between A and B, be taken as the



origin of coordinates, CB, produced indefinitely, being the axis of x. From C draw CR at right angles to PQ.

Let
$AC = c = BC$, $\angle PAB = \theta$, $\angle RCB = \psi$, $CR = \delta$, $AP = a = BQ$.

Then the equation to PQ is

$$x\cos\psi + y\sin\psi = \delta = \tfrac{1}{2}a\{\cos(\psi - \theta) + \sin(\psi - \theta)\},$$

also $$2c\cos\psi = a\{\cos(\psi - \theta) - \sin(\psi - \theta)\}.$$

Hence, squaring and adding these equations, we have

$$4(x\cos\psi + y\sin\psi)^2 + 4c^2\cos^2\psi = 2a^2 \ldots\ldots\ldots\ldots(1),$$

and therefore

$$(x\cos\psi + y\sin\psi)^2 = \frac{a^2 - 2c^2}{2}\cos^2\psi + \frac{a^2}{2}\sin^2\psi.$$

This equation represents a straight line always touching a conic section $$\frac{x^2}{a^2 - 2c^2} + \frac{y^2}{a^2} = \frac{1}{2}.$$

Cor. If $2c^2 = a^2$, the equation (1) becomes

$$(x\cos\psi + y\sin\psi)^2 = c^2\sin^2\psi,$$

which shews that PQ always passes through a point

$$x = 0, \quad y = c.$$

2. To find the envelop of a right line such that the product of perpendiculars drawn to it from two given points may be constant.

Lemma. If η, θ, denote the reciprocals of the intercepts of the axes of any conic section from the centre made by any tangent, then $$a^2\eta^2 \pm b^2\theta^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

For, let x', y', be the coordinates of the point of contact: then, the equation to the conic section being

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

we shall have for the equation to the tangent,

$$\frac{xx'}{a^2} \pm \frac{yy'}{b^2} = 1,$$

and therefore $$\frac{x'}{a} = a\eta, \quad \pm\frac{y'}{b} = b\theta.$$

Hence, by the equation to the curve,

$$a^2\eta^2 \pm b^2\theta^2 = 1.$$

The equation (1) has been called by Mr. Booth, in an ingenious treatise entitled "*On the Application of a New Analytic Method to the Theory of Curves and Curved Surfaces*," the *tangential equation* to the conic section. The *general tangential equation* of a conic section, given in his work, is of the form

$$a\eta^2 + b\theta^2 + 2c\eta\theta + 2a'\eta + 2b'\theta = 1,$$

c being zero in the case of a parabola.

In the present problem, let the line joining the two given points be chosen as the axis of x, its middle point being taken as the origin of coordinates. The equation to the moveable right line is of the form

$$\eta x + \theta y = 1.$$

Let a represent the distance of either of the given points from the origin, and δ, δ', their distances from the moveable line. Then

$$\delta = \pm \frac{1 - \eta a}{(\eta^2 + \theta^2)^{\frac{1}{2}}}, \quad \delta' = \pm \frac{1 + \eta a}{(\eta^2 + \theta^2)^{\frac{1}{2}}},$$

and therefore, c^2 being taken to represent the constant value of $\delta\delta'$, we have, as the tangential equation to the envelop,

$$\pm c^2 = \frac{1 - \eta^2 a^2}{\eta^2 + \theta^2},$$

$$(a^2 \pm c^2)\,\eta^2 \pm c^2\theta^2 = 1.$$

Comparing this equation with (1) we see that the required envelop is a conic section the semi-axes of which are $(a^2 \pm c^2)^{\frac{1}{2}}$ and c.

For other exemplifications of the *Method of Tangential Coordinates* the student is referred to Mr. Booth's treatise.

3. To find the curve, which is touched by all the chords of a central conic section, which subtend a right angle at the centre.

If the conic section be represented by the equation

$$ax^2 + by^2 = c,$$

the required locus will be a circle concentric with the ellipse and having a radius equal to

$$\left(\frac{c}{a+b}\right)^{\frac{1}{2}}.$$

Querret: *Gergonne, Annales de Mathématiques*, tom. XV. p. 197.
Un Abonné: *Ibid.*, tom. XV. p. 199.

4. Through the centre of a conic section are drawn two straight lines at right angles to each other, one meeting the curve in P, the other the minor director in R: to prove that the line PR envelops a circle the diameter of which is, in the case of the ellipse, the minor axis, and, in that of the hyperbola, the transverse axis.

Booth: *Annals of Philosophy*, 1845, vol. II. p. 541.

5. A given angle moves along a given right line, one side always passing through a given point; to find the equation to the curve which is always touched by the other side.

The given right line being taken for the axis of x, and a perpendicular to the given right line, through the given point, as that of y, then, the ordinate of the given point being β, and the given angle being denoted by $\tan^{-1} m$, the equation to the required curve will be

$$(y + mx - \beta)^2 = 4m\beta\,(my - x),$$

which belongs to a parabola.

Booth: *Application of a New Analytic Method to the Theory of Curves and Curved Surfaces*, p. 9.

6. The vertex of a right angle moves along a given circle, and one side passes through a fixed point: to find the envelop of the other side.

Let the centre of the circle be chosen as the origin of coordinates, the axis of x passing through the fixed point: let c represent the distance of the fixed point from the centre of the circle, a the radius of the circle.

Then the required envelop will be a conic section of which the tangential equation is

$$a^2\eta^2 + (a^2 - c^2)\,\theta^2 = 1.$$

Booth: *Application of a New Analytic Method to the Theory of Curves and Curved Surfaces*, p. 5.

7. To find the envelop of the conic section

$$lvw + mwu + nuv = 0,$$

u, v, w, being given linear functions of x and y, and l, m, n, being arbitrary parameters subject to the following relation,

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0.$$

The required envelop is the straight line

$$\alpha u + \beta v + \gamma w = 0.$$

Hearn: *Researches on Curves of the Second Order*, p. 34.

8. To find the envelop of the conic section

$$(lu)^{\frac{1}{2}} + (mv)^{\frac{1}{2}} + (nw)^{\frac{1}{2}} = 0,$$

u, v, w, being given linear functions of x and y, and l, m, n, denoting arbitrary parameters subject to the following relation, viz.

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0.$$

The required envelop is the straight line defined by the equation

$$\alpha u + \beta v + \gamma w = 0.$$

Hearn: *Researches on Curves of the Second Order*, p. 36.

9. To find the envelop of the base of a triangle inscribed in a conic section, two sides of which pass through fixed points.

Let $w = 0$ be the equation to the chord passing through the two fixed points; $u = 0$, $v = 0$, those to the tangents at its extremity. Then, the equations to the lines joining the fixed points to the intersection of $u = 0$, $v = 0$, being

$$au = v, \quad bu = v,$$

the required envelop is a conic section represented by the equation

$$uv = \frac{(a+b)^2}{4ab} \cdot w^2.$$

Salmon: *Treatise on Conic Sections*, p. 222.

For additional examples on the subject of conical envelops the student is referred to Mr. Salmon's *Treatise on Conic Sections.*

10. Conic sections are inscribed in the same quadrilateral: to prove that the polar of any point in their plane envelops a conic section.

Booth: *On the Application of a New Analytic Method to the Theory of Curves and Curved Surfaces*, p. 6.

SECTION XXVII.

Similar Curves.

1. To prove that two general equations of the second order

$$ax^2 + 2cxy + by^2 + \ldots\ldots = 0,$$

$$a_{\prime}x^2 + 2c_{\prime}xy + b_{\prime}y^2 + \ldots\ldots = 0$$

will represent similar curves, if

$$\frac{c^2 - ab}{(a+b)^2} = \frac{c_{\prime}^2 - a_{\prime}b_{\prime}}{(a_{\prime} + b_{\prime})^2}.$$

Changing the axes of the second curve through an angle θ, its equation will become

$$a_{\prime}(x'\cos\theta - y'\sin\theta)^2 + 2c_{\prime}(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$
$$+ b_{\prime}(x'\sin\theta + y'\cos\theta)^2 + \ldots\ldots = 0;$$

and therefore, in order that the two curves may be similar in form, θ being supposed to be of such a magnitude that they are similarly situated in regard to their coordinate axes, we must have, λ being some arbitrary quantity,

$$2\lambda a = a_{,} + b_{,} + (a_{,} - b_{,}) \cos 2\theta + 2c_{,} \sin 2\theta \ldots\ldots (1),$$

$$2\lambda b = a_{,} + b_{,} - (a_{,} - b_{,}) \cos 2\theta - 2c_{,} \sin 2\theta \ldots\ldots (2),$$

$$2\lambda c = - (a_{,} - b_{,}) \sin 2\theta + 2c_{,} \cos 2\theta \ldots\ldots\ldots\ldots\ldots (3).$$

Hence we have

$$(1) + (2), \ldots\ldots \lambda (a + b) = a_{,} + b_{,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$(1) - (2), \ldots\ldots \lambda (a - b) = (a_{,} - b_{,}) \cos 2\theta + 2c_{,} \sin 2\theta \ldots (5),$$

$$(3)^2 + (5)^2, \ldots\ldots \lambda^2 \{4c^2 + (a - b)^2\} = (a_{,} - b_{,})^2 + 4c_{,}^2 \ldots\ldots (6),$$

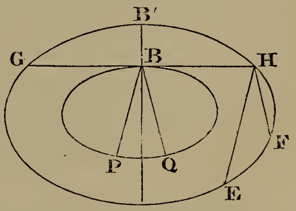
$$(6) - (4)^2, \ldots\ldots \lambda^2 (c^2 - ab) = c_{,}^2 - a_{,}b_{,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$(7) \div (4)^2, \ldots\ldots \frac{c^2 - ab}{(a + b)^2} = \frac{c_{,}^2 - a_{,}b_{,}}{(a_{,} + b_{,})^2}.$$

This condition of similarity is ascribed by Mr. Salmon to Mr. Jellett.

Salmon: *Treatise on Conic Sections*, p. 194.

2. *GH* is the chord of an ellipse, which touches a concentric, similar, and similarly placed, ellipse, at the extremity *B* of one of its axes. From *B* are drawn any two chords *BP*, *BQ*, in the smaller ellipse, equally inclined to the axis through *B*, and from *H* are drawn, in the larger ellipse, two chords *HE*, *HF*, parallel respectively to *BP*, *BQ*. To prove that, denoting *BP* or *BQ* by r, *HE* by r', and *HF* by r'',



$$r' + r'' = 2r.$$

Let θ denote the inclination of the chord *BP* to the axis through *B*; then, $2b$ denoting this axis, and $2a$ the other, we have by the polar equation,

$$\left(1 - \frac{a^2 - b^2}{a^2} \sin^2\theta\right) r = 2b \cos\theta \ldots\ldots\ldots\ldots (1).$$

Again, by the polar equation to the larger ellipse, *H* being the pole, $2a_{,,}$, $2b_{,,}$ denoting its axes, and h, k, the distances *BB′*, *BH*,

$$\left(1 - \frac{a_{,}^2 - b_{,}^2}{a_{,}^2} \sin^2\theta\right) r' = 2 (b_{,} - h) \cos\theta + 2k \frac{b_{,}^2}{a_{,}^2} \sin\theta,$$

or, the two ellipses being similar, and $b_{,} - h$ being equal to b,

$$\left(1 - \frac{a^2 - b^2}{a^2}\sin^2\theta\right) r' = 2b\cos\theta + 2k\frac{b^2}{a^2}\sin\theta\ldots\ldots(2).$$

Again, putting $-\theta$ for θ and r'' for r', we have

$$\left(1 - \frac{a^2 - b^2}{a^2}\sin^2\theta\right) r'' = 2b\cos\theta - 2k\frac{b^2}{a^2}\sin\theta\ldots\ldots(3).$$

From (2) and (3) we have

$$\left(1 - \frac{a^2 - b^2}{a^2}\sin^2\theta\right)(r' + r'') = 4b\cos\theta,$$

and therefore, by (1), $\quad r' + r'' = 2r.$

This proposition is due to Clairaut and serves as the basis of his investigations respecting the figure of the planets on the hypothesis of homogeneity of substance.

Clairaut: *Théorie de la Figure de la Terre*, p. 158.
Puissant: *Recueil de diverses propositions de Géométrie*, p. 156, *troisième édition.*

3. To prove that two curves of the second order, not of the same kind, cannot be similar.

De la Hire: *Sectiones Conicæ*, lib. VI. prop. 1.

To prove that all parabolas are similar curves.

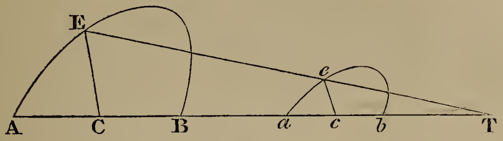
De la Hire: *Sectiones Conicæ*, lib. VI. prop. 3.

4. If the inclination of a system of parallel chords in one central conic section to their diameter be the same as that of similarly related chords and diameter in another central conic section, and if these diameters are to each other as their parameters, to prove that the two conic sections are similar.

De la Hire: *Sectiones Conicæ*, lib. VI. prop. 2.

5. Two similar unequal ellipses AEB, aeb, have similar diameters AB, ab, and centres C, c, in the same straight line AT, the two conjugate diameters being parallel. From the centres C, c, are drawn the semi-diameters CE, ce, parallel

to each other; E, e, are joined, and Ee is produced to meet Cc,



produced, in T. To shew that

$$CT : cT :: EC : ec.$$

De la Hire: *Sectiones Conicæ*, lib. VI. prop. 6.

6. To shew that all hyperbolas, which have the same asymptotes, are similar curves.

De la Hire: *Sectiones Conicæ*, lib. VI. prob. 8.

7. If, through the common summit A of two similar curves of the second order, the greater axes of which coincide in direction, any two chords are drawn cutting one of the curves in B, B', and the other in D, D', to shew that

$$AB : AB' :: AD : AD'.$$

Puissant: *Recueil de diverses propositions de Géométrie*, p. 154, *troisième édition.*

8. Any number of concentric, similar, and similarly situated ellipses or hyperbolas, are intersected by a line parallel to a directrix in P, P', P'',...: to prove that the extremities of diameters respectively conjugate to the diameters through P, P', P'',... are in a line perpendicular to the directrix.

9. Any number of ellipses, concentric, similar, and similarly placed, are intersected by an hyperbola

$$xy = c^2,$$

in points P, P', P'',...: to prove that the extremities of the diameters respectively conjugate to the diameters through P, P', P'',.. are situated in an hyperbola

$$xy = -c^2.$$

10. Two similar ellipses have a common vertex, the directions of their major axes being opposite; a common tangent meets them in P, Q, and a perpendicular to their major axes through the common vertex meets PQ in O: to prove that $OP = OQ$.

SECTION XXVIII.

Miscellaneous Problems.

1. To find the locus of the summit of a moveable angle, of invariable magnitude, the two sides of which always touch a conic section.

By properly choosing the axes, the equation to the conic section may be written in the form

$$y^2 + ax^2 + bx = 0 \dots\dots\dots\dots\dots\dots (1).$$

Let h, k, be the coordinates of the intersection of the two sides of the angle in any position: then the equation to either of its sides will be

$$y - k = m(x - h) \dots\dots\dots\dots\dots\dots (2).$$

Eliminating y between (1) and (2), we shall obtain a quadratic in x, the two roots of which must be equal: this condition will, after performing the requisite reductions, give us the equation

$$4(ah^2 + bh)m^2 - 4(2ahk + bk)m + (4ak^2 - b^2) = 0.$$

Representing the two roots of this equation by m, m', we shall have

$$m + m' = \frac{2ahk + bk}{ah^2 + bh}, \qquad 4mm' = \frac{4ak^2 - b^2}{ah^2 + bh},$$

and therefore $$(m - m')^2 = \frac{b^2(k^2 + ah^2 + bh)}{(ah^2 + bh)^2},$$

and $$1 + mm' = \frac{4a(h^2 + k^2) + 4bh - b^2}{4(ah^2 + bh)}.$$

But, the magnitude of the moveable angle being constant,

$$m - m' = \lambda (1 + mm'),$$

where λ represents a constant quantity. Hence, eliminating m and m' from the last three equations, we have, for the equation to the required locus,

$$16b^2 (k^2 + ah^2 + bh) = \lambda^2 \{4a (h^2 + k^2) + 4bh - b^2\}^2.$$

Poncelet: *Gergonne, Annales de Mathématiques*, tom. VIII. p. 201.

2. To find the locus of the middle points of a system of chords in a conic section, which all pass through a given point.

Let a diameter through the given point, as origin, be taken as the axis of x, and a straight line, parallel to its conjugate, as the axis of y.

The equation to the curve will then be of the form

$$y^2 = ax^2 + 2bx + c \dots\dots\dots\dots (1).$$

The equation to one of the system of chords will be

$$y = mx \dots\dots\dots\dots (2).$$

At the intersections of (1) and (2), there is

$$(a - m^2) x^2 + 2bx + c = 0.$$

The abscissa x' of the middle point of (2) will be equal to the semi-sum of the roots of this equation; hence

$$x' = \frac{b}{m^2 - a} \dots\dots\dots\dots (3).$$

Also, y' being the ordinate of the middle point,

$$y' = mx' \dots\dots\dots\dots (4).$$

Eliminating m between (3) and (4), we obtain for the equation to the required locus,

$$y'^2 = ax'^2 + bx',$$

which is therefore a conic section similar to the proposed one and similarly situated, passing through its centre and through the given point.

Durrande: *Gergonne, Annales de Mathématiques*, tom. IX. p. 122.

3. To determine the positions of the foci of a conic section of which the equation is given.

The distance of the focus (α, β) of a conic section, from any point of the curve, is always a linear function of the coordinates x, y, of the point.

Let the equation to any conic section, referred to a system of rectangular coordinates, be

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0 \ldots\ldots\ldots (1).$$

Then, by the nature of the focus,

$$\{(x-\alpha)^2 + (y-\beta)^2\}^{\frac{1}{2}} = gx + hy + k \ldots\ldots\ldots (2),$$

where g, h, k, are constants.

Since the equations (1) and (2) must be identical, we have, making use of an arbitrary multiplier λ,

$$(x-\alpha)^2+(y-\beta)^2-(gx+hy+k)^2=\lambda(ax^2+by^2+2cxy+2a'x+2b'y+c'),$$

and, equating the coefficients of like variables, we get six equations,

$$\begin{aligned} 1 - g^2 &= \lambda a, & -\alpha - kg &= \lambda a', \\ 1 - h^2 &= \lambda b, & -\beta - kh &= \lambda b', \\ -gh &= \lambda c, & \alpha^2 + \beta^2 - k^2 &= \lambda c', \end{aligned}$$

involving the six unknown quantities α, β, g, h, k, λ.

If we eliminate g, h, k, λ, we shall obtain two equations of the second degree in α, β.

From these two equations we may obtain four pairs of values of (α, β): two only of the pairs, however, are possible.

Ex. Take the case of the ellipse referred to its axes, when its equation will be

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1.$$

Then $a = \dfrac{1}{A^2}$, $b = \dfrac{1}{B^2}$, $c = 0$, $a' = 0$, $b' = 0$, $c' = -1$:

and the six equations become

$$1 - g^2 = \frac{\lambda}{A^2}, \qquad \alpha + kg = 0,$$

$$1 - h^2 = \frac{\lambda}{B^2}, \qquad \beta + kh = 0,$$

$$gh = 0, \qquad \alpha^2 + \beta^2 - k^2 = -\lambda.$$

Hence we obtain two systems of values,

$$g = 0, \qquad h = 0,$$

$$h = \frac{1}{B}(B^2 - A^2)^{\frac{1}{2}}, \qquad g = \frac{1}{A}(A^2 - B^2)^{\frac{1}{2}},$$

$$k = B, \qquad k = A,$$

$$\alpha = 0, \qquad \beta = 0,$$

$$\beta = -(B^2 - A^2)^{\frac{1}{2}}. \qquad \alpha = -(A^2 - B^2)^{\frac{1}{2}}.$$

Since A and B have different values, it follows that one, and one only, of these systems of equations will give possible values to α and β.

Bret: *Gergonne, Annales de Mathématiques*, tom. VIII. p. 317.

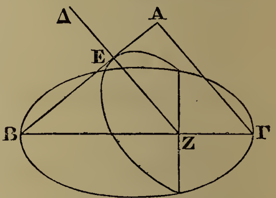
4. To prove that the difference between the squares of the reciprocals of any two coincident semi-diameters of two conic sections, which have the same minor directors, is constant.

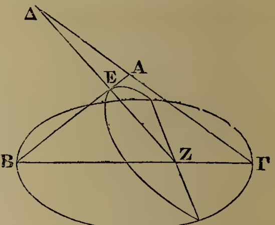
Booth: *Annals of Philosophy*, 1845, vol. II. p. 545.

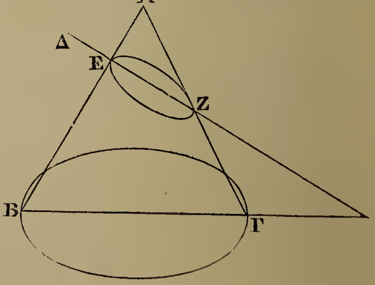
APPENDIX.

The etymology of the names Parabola, Ellipse, Hyperbola, is given by Eutochius in his Commentary on the first book of the Conics of Apollonius.

Ὁ δὲ λέγει σαφὲς ποιήσομεν ἐπὶ τῶν ὑποκειμένων καταγραφῶν. ἔστω τὸ διὰ τοῦ ἄξονος κώνου τρίγωνον τὸ ΑΒΓ, καὶ ἤχθω τῇ ΑΒ ἀπὸ τυχόντος σημείου τοῦ Ε πρὸς ὀρθὰς ἡ ΔΕΖ, καὶ τότε διὰ τὸ ΔΖ ἐπίπεδον ἐμβληθὲν ὀρθὸν πρὸς τῷ ΑΒ τεμνέτω τὸν κῶνον· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΑΕΔ, ΑΕΖ, γωνιῶν, καὶ ὀρθογωνίου μένοντος τοῦ κώνου, καὶ ὀρθῆς δηλονότι τῆς ὑπὸ ΒΑΓ γωνίας, ὡς ἐπὶ τῆς πρώτης καταγραφῆς, δύο ὀρθαὶ ἔσονται αἱ ὑπὸ ΒΑΓ, ΑΕΖ, γωνίαι. ὥστε παράλληλός ἐστιν ἡ ΔΕΖ τῇ ΑΓ, καὶ γίνεται ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τομὴ ἡ καλουμένη Παραβολὴ, οὕτω κληθεῖσα ἀπὸ τοῦ παράλληλον εἶναι τὸ ΔΕΖ, ἥτις ἐστὶ κοινὴ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου, τῇ ΑΓ πλευρᾷ τοῦ τριγώνου. ἐὰν δὲ ἀμβλυγώνιος ᾖ ὁ κῶνος, ὡς ἐπὶ τῆς δευτέρας καταγραφῆς, ἀμβλείας δηλονότι οὔσης τῆς ὑπὸ ΒΑΓ, ὀρθῆς δὲ τῆς ὑπὸ ΑΕΖ, δύο ὀρθῶν μείζους ἔσονται αἱ ὑπὸ ΒΑΓ, ΑΕΖ, γωνίαι· ὥστε οὐ συμ-







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